Linear ODEs

For a **linear ODE of order 2** we have an initial value problem

$$y'' + a_1(t)y' + a_0(t)y = f(t), \qquad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

For a **linear ODE of order** n we have an initial value problem

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_0(t)y = f(t), \qquad y(t_0) = y_0,\ y'(t_0) = y_0', \ldots, y^{(n-1)}(t_0) = y_0^{(n-1)}$$

We call the ODE **homogeneous** if $f(t) = 0$, otherwise **inhomogeneous**.
We have the following existence and uniqueness result:

Theorem 1. *Assume that the functions $a_0(t), \ldots, a_{n-1}(t)$ and $f(t)$ are continuous for $t \in (a,b)$. Then the initial value problem has a unique solution $y(t)$ for $t \in (a,b)$.*

We will first consider ODEs of order 2. Everything will carry over to the general case.
Let us define the "differential operator" L as

$$Ly = y'' + a_1(t)y' + a_0(t)y$$

The operator L is **linear**:

- for $c \in \mathbb{R}$ we have $L(cy) = cLy$
- for functions $u(t), v(t)$ we have $L(u+v) = Lu + Lv$

Homogeneous linear ODE of order 2

We have the initial value problem

$$y'' + a_1(t)y' + a_0(t)y = 0, \qquad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

- Find two solutions $Y_1(t), Y_2(t)$ of the homogeneous ODE.
  (We will explain how to do this for constant coefficients, but there is no systematic recipe in the general case.)
  Then for any $c_1, c_2 \in \mathbb{R}$ the function
  $$\boxed{y(t) = c_1Y_1(t) + c_2Y_2(t)} \tag{1}$$
  is a solution of $Ly = 0$. But can we use this $y(t)$ to solve our initial value problem?
- For the inital conditions we get the linear system
  $$\underbrace{\begin{bmatrix} Y_1(t_0) & Y_2(t_0) \\ Y_1'(t_0) & Y_2'(t_0) \end{bmatrix}}_{A(t_0)} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$
- If the matrix $A(t_0)$ is **nonsingular**: we get a solution $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ for any given initial data $\begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$. Hence any solution of $Ly = 0$ has the form (1) (since the IVP has a unique solution). We call $Y_1(t), Y_2(t)$ a **fundamental set of solutions**.
- If the matrix $A(t_0)$ is **singular**: there exists $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ such that $A(t_0)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This means that $y(t) = c_1Y_1(t) + c_2Y_2(t)$ is a solution of the ODE with $y(t_0) = 0$ and $y'(t_0) = 0$. But the constant function $y(t) = 0$ is a solution of this IVP. By the uniqueness we must have
  $$c_1Y_1(t) + c_2Y_2(t) = 0 \qquad \text{for all } t \in (a,b)$$
  This means that **the functions $Y_1(t)$ and $Y_2(t)$ are linearly dependent**. By taking the derivative we get $c_1Y_1'(t) + c_2Y_2'(t) = 0$ for $t \in (a,b)$, i.e.,
  $$\underbrace{\begin{bmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{bmatrix}}_{A(t)} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{for all } t \in (a,b)$$
  so the matrix $A(t)$ is singular for all $t \in (a,b)$.

Inhomogeneous linear ODE of order 2

We have the initial value problem

$$y'' + a_1(t)y' + a_0(t)y = f(t), \qquad y(t_0) = y_0, \quad y'(t_0) = y'_0$$

- First find a fundamental set of solutions $Y_1(t), Y_2(t)$ for the homogeneous problem $Ly = 0$.
- Find one solution Y ("particular solution") satisfying $LY = f$.
  We will explain how to do this if we have $Y_1(t), Y_2(t)$. In the case of constant coefficients there is the easier "method of undetermined coefficients" for certain functions $f(t)$.
- a solution $y(t)$ of the ODE satisfies $Ly = f$. Subtracting $LY = f$ gives $L(y - Y) = 0$, i.e., $y(t) - Y(t)$ is a solution of the homogeneous ODE, hence $y - Y = c_1Y_1 + c_2Y_2$. Therefore the general solution of the ODE is

$$\boxed{y(t) = Y(t) + c_1Y_1(t) + c_2Y_2(t)}$$

- For the inital conditions we get the linear system

$$\underbrace{\begin{bmatrix} Y_1(t_0) & Y_2(t_0) \\ Y'_1(t_0) & Y'_2(t_0) \end{bmatrix}}_{A(t_0)} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 - Y(t_0) \\ y'_0 - Y'(t_0) \end{bmatrix}$$

  We know that this matrix is nonsingular, so we can solve this linear system and obtain the solution of the initial value problem.

Higher order ODEs

E.g., for a linear ODE of order 3 we have the initial value problem

$$y''' + a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t), \qquad y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad y''(t_0) = y''_0$$

- find solutions $Y_1(t), Y_2(t), Y_3(t)$ of the homogeneous problem $Ly = 0$.
- for the initial conditions we get a linear system with the matrix

$$A(t_0) = \begin{bmatrix} Y_1(t_0) & Y_2(t_0) & Y_3(t_0) \\ Y'_1(t_0) & Y'_2(t_0) & Y'_3(t_0) \\ Y''_1(t_0) & Y''_2(t_0) & Y''_3(t_0) \end{bmatrix}$$

- **If the matrix** $A(t_0)$ **is nonsingular** we call $Y_1(t), Y_2(t), Y_3(t)$ a **fundamental set of solutions**. The **general solution of the homogeneous ODE** $Ly = 0$ is given by

$$\boxed{y(t) = c_1Y_1(t) + c_2Y_2(t) + c_3Y_3(t)}$$

  If the matrix $A(t_0)$ **is singular**, there exists $\vec{c} \neq \vec{0}$ with $A(t_0)\vec{c} = \vec{0}$. Hence $c_1Y_1(t) + c_2Y_2(t) + c_3Y_3(t) = 0$ for all $t \in (a, b)$, i.e., the functions Y_1, Y_2, Y_3 are linearly dependent. We also get $A(t)\vec{c} = \vec{0}$ for all $t \in (a, b)$, i.e., the matrix $A(t)$ is singular for all $t \in (a, b)$.
- For the inhomogeneous ODE find a particular solution Y satisfying $LY = f$. Then the **general solution of the inhomogeneous ODE** $Ly = f$ is given by

$$\boxed{y(t) = Y(t) + c_1Y_1(t) + c_2Y_2(t) + c_3Y_3(t)}$$