Linear system of ODEs

We want to find n functions $y_1(t),\dots,y_n(t)$ satisfying n differential equations which depend linearly on $y_j(t)$.
We first consider the case $n=2$. The general case is completely analogous.

We have the following **initial value problem**: Find $\vec{y}(t)=\begin{bmatrix} y_1(t)\\ y_2(t)\end{bmatrix}$ such that

$$\begin{bmatrix} y_1'(t)\\ y_2'(t)\end{bmatrix}=\begin{bmatrix} a_{11}(t) & a_{12}(t)\\ a_{21}(t) & a_{22}(t)\end{bmatrix}\begin{bmatrix} y_1(t)\\ y_2(t)\end{bmatrix}+\begin{bmatrix} f_1(t)\\ f_2(t)\end{bmatrix},\qquad \begin{bmatrix} y_1(t_0)\\ y_2(t_0)\end{bmatrix}=\begin{bmatrix} y_1^{(0)}\\ y_2^{(0)}\end{bmatrix}$$

$$\vec{y}\,'(t)=A(t)\vec{y}(t)+\vec{f}(t),\qquad \vec{y}(t_0)=\vec{y}^{(0)}$$

We call the ODE **homogeneous** if $\vec{f}(t)=\vec{0}$, otherwise **inhomogeneous**.
We have the following **existence and uniqueness** result:

Theorem 1. *Assume that the functions $a_{jk}(t)$ and $f_j(t)$ are continuous for $t\in(\alpha,\beta)$. Let $t_0\in(\alpha,\beta)$. Then the initial value problem has a unique solution $\vec{y}(t)$ for $t\in(\alpha,\beta)$.*

Homogeneous case

We have the initial value problem

$$\begin{bmatrix} y_1'(t)\\ y_2'(t)\end{bmatrix}=\begin{bmatrix} a_{11}(t) & a_{12}(t)\\ a_{21}(t) & a_{22}(t)\end{bmatrix}\begin{bmatrix} y_1(t)\\ y_2(t)\end{bmatrix},\qquad \begin{bmatrix} y_1(t_0)\\ y_2(t_0)\end{bmatrix}=\begin{bmatrix} y_1^{(0)}\\ y_2^{(0)}\end{bmatrix}$$

- Find two solutions $\vec{Y}^{(1)}(t),\vec{Y}^{(2)}(t)$ of the homogeneous ODE $\vec{y}\,'=A\vec{y}$.
  (We will explain how to do this for constant coefficients, but there is no systematic recipe in the general case.)
  Then for any $c_1,c_2\in\mathbb{R}$ the function

  $$\boxed{\vec{y}(t)=c_1\vec{Y}^{(1)}(t)+c_2\vec{Y}^{(2)}(t)}\tag{1}$$

  is a solution of $\vec{y}\,'=A\vec{y}$. But can we use this $\vec{y}(t)$ to solve our initial value problem?
  Consider the 2×2 matrix $\Psi(t):=\left[\vec{Y}^{(1)}(t),\vec{Y}^{(2)}(t)\right]$.
- For the inital conditions we get the 2×2 linear system $c_1\vec{Y}^{(1)}(t_0)+c_2\vec{Y}^{(2)}(t_0)=\vec{y}^{(0)}$

  $$\underbrace{\left[\vec{Y}^{(1)}(t_0)\,,\,\vec{Y}^{(2)}(t_0)\right]}_{\Psi(t_0)}\begin{bmatrix} c_1\\ c_2\end{bmatrix}=\vec{y}^{(0)}$$

- **If the matrix $\Psi(t_0)$ is nonsingular:** we get a solution $\begin{bmatrix} c_1\\ c_2\end{bmatrix}$ for any given initial data $\vec{y}^{(0)}$. Hence any solution of $\vec{y}\,'=A\vec{y}$ has the form (1) (since the IVP has a unique solution). We call $\vec{Y}^{(1)}(t),\vec{Y}^{(2)}(t)$ a **fundamental set of solutions**.
- **If the matrix $\Psi(t_0)$ is singular:** there exists $\begin{bmatrix} c_1\\ c_2\end{bmatrix}\neq\begin{bmatrix} 0\\ 0\end{bmatrix}$ such that $\Psi(t_0)\begin{bmatrix} c_1\\ c_2\end{bmatrix}=\begin{bmatrix} 0\\ 0\end{bmatrix}$. This means that $\vec{y}(t)=c_1\vec{Y}^{(1)}(t)+c_2\vec{Y}^{(2)}(t)$ is a solution of the ODE with $\vec{y}(t_0)=\vec{0}$. But the constant function $\vec{y}(t)=\vec{0}$ is a solution of this IVP. By the uniqueness we must have

  $$c_1\vec{Y}^{(1)}(t)+c_2\vec{Y}^{(2)}(t)=\vec{0}\qquad\text{for all } t\in(\alpha,\beta)$$

  $$\underbrace{\left[\vec{Y}^{(1)}(t)\,,\,\vec{Y}^{(2)}(t)\right]}_{\Psi(t)}\begin{bmatrix} c_1\\ c_2\end{bmatrix}=\vec{0}$$

  This means that **the functions $\vec{Y}^{(1)}(t)$ and $\vec{Y}^{(2)}(t)$ are linearly dependent**, and the matrix $\Psi(t)$ is singular for all $t\in(\alpha,\beta)$.

Inhomogeneous case

We have the initial value problem

$$\vec{y}'(t) = A(t)\vec{y}(t) + \vec{f}(t), \qquad \vec{y}(t_0) = \vec{y}^{(0)}$$

- First find a fundamental set of solutions $\vec{Y}^{(1)}(t), \vec{Y}^{(2)}(t)$ for the homogeneous problem $\vec{y}' = A\vec{y}$.
- Find one solution $\vec{Y}$ ("particular solution") satisfying $\vec{y}' = A\vec{y} + \vec{f}$.
  We will explain how to do this ("variation of parameters") if we have a fundamental set $\vec{Y}^{(1)}(t), \vec{Y}^{(2)}(t)$. In the case of constant coefficients there is the easier "method of undetermined coefficients" for certain functions $\vec{f}(t)$.
- a solution $\vec{y}(t)$ of the ODE satisfies $\vec{y}' = A\vec{y} + \vec{f}$. Subtracting $\vec{Y}' = A\vec{Y} + \vec{f}$ gives $(\vec{y} - \vec{Y})' = A(\vec{y} - \vec{Y})$, i.e., $\vec{y}(t) - \vec{Y}(t)$ is a solution of the homogeneous ODE. Hence $\vec{y} - \vec{Y} = c_1\vec{Y}^{(1)} + c_2\vec{Y}^{(2)}$. Therefore the **general solution of the ODE** is

  $$\boxed{\vec{y}(t) = \vec{Y}(t) + c_1\vec{Y}^{(1)}(t) + c_2\vec{Y}^{(2)}(t)}$$

- For the inital conditions we get the linear system

  $$\underbrace{\left[\vec{Y}^{(1)}(t_0)\ ,\ \vec{Y}^{(2)}(t_0)\right]}_{\Psi(t_0)} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{y}^{(0)} - \vec{Y}(t_0)$$

  We know that the matrix $\Psi(t_0)$ is nonsingular, so we can solve this linear system and obtain the solution of the initial value problem.