Recipe: using the Laplace transform for piecewise functions

We want to solve the initial value problem

$$y'' + a_1 y' + a_0 y = f, \qquad y(0) = y_0, \quad y'(0) = y_0'$$

with a **piecewise defined forcing function**

$$f(t) = \begin{cases} f_1(t) & \text{for } t \in [0, c) \\ f_2(t) & \text{for } t \in [c, \infty) \end{cases}$$

(1) Rewrite $f(t)$ using the unit step function $u(t)$ (a.k.a. Heaviside function):

$$f(t) = f_1(t) + \boxed{u(t-c)g(t)} \quad \text{with } \boxed{g(t) := f_2(t) - f_1(t)}$$

(2) Find the Laplace transform $F = \mathscr{L}[f]$:

$$F(s) = F_1(s) + \boxed{e^{-cs}\mathscr{L}[g(t+c)]} \quad \longleftarrow \boxed{\text{Don't forget the } \textbf{shift } g(t+c) \textbf{ before} \text{ applying } \mathscr{L}}$$

(3) Take the Laplace transform of left hand side of the ODE: Let

$$Y = \mathscr{L}[y], \qquad Y_1 = \mathscr{L}[y'] = s \cdot Y - y(0), \qquad Y_2 = \mathscr{L}[y''] = s \cdot Y_1 - y'(0)$$

and solve the following equation for $Y(s)$:

$$Y_2 + a_1 Y_1 + a_0 Y = F$$

yielding

$$Y(s) = R_1(s) + e^{-cs} R_2(s)$$

with rational functions $R_1(s)$, $R_2(s)$.

(4) Find the inverse Laplace transform $y = \mathscr{L}^{-1}[Y]$:
Find the partial fraction decompositions of $R_1(s)$ and $R_2(s)$.
Find $r_1(t) = \mathscr{L}^{-1}[R_1(s)]$, $r_2(t) = \mathscr{L}^{-1}[R_2(s)]$, then

$$y(t) = r_1(t) + \boxed{u_c(t) r_2(t-c)} \quad \longleftarrow \boxed{\text{Don't forget the } \textbf{shift } r_2(t-c) \textbf{ after} \text{ applying } \mathscr{L}^{-1}}$$