Linear ODE with constant coefficients: find particular solution using "Method of undetermined coefficients"

We have the inhomogeneous ODE

$$\underbrace{y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y}_{Ly} = 0$$

with constants $a_0, \ldots, a_{n-1} \in \mathbb{R}$.

1. **solve homogeneous ODE** $Ly = 0$:
   find roots $r_1, \ldots, r_n \in \mathbb{C}$ **of characteristic equation** $r^n + a_{n-1}r^{n-1} + \cdots + a_0 = 0$
   for real root r of multiplicity m use $e^{rt}, \ldots, t^{m-1}e^{rt}$
   for complex conjugate roots $\alpha + i\beta$, $\alpha - i\beta$ of multiplicity m
   use $e^{\alpha t}\cos(\beta t), \ldots, t^{m-1}e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t), \ldots, t^{m-1}e^{\alpha t}\sin(\beta t)$
   this gives linearly independent functions $Y_1(t), \ldots, Y_n(t)$
2. **find particular solution** $Y(t)$ satisfying $LY = f$

If the function $f(t)$ **has a** $\boxed{\textbf{special form}}$ (**see cases below**) we can use the "**method of undetermined coefficients**":

- try a particular solution $Y(t)$ of the same form, but with unknown coefficients
- plug this into $LY = f$,
- compare terms on both sides, solve for the unknown coefficients

Recall the **key identities** (obtained by repeatedly applying ∂_r)

$$Le^{rt} = p(r)e^{rt} \tag{1}$$

$$L(te^{rt}) = p'(r)e^{rt} + p(r)te^{rt} \tag{2}$$

$$L(t^2e^{rt}) = p''(r)e^{rt} + 2p'(r)te^{rt} + p(r)t^2e^{rt} \tag{3}$$

Case 1: exponential function $\boxed{f(t) = \alpha_0 e^{\mu t}}$

Try $\boxed{f(t) = A_0 e^{\mu t}}$ with unknown coefficient A_0
this works if μ is NOT a root of the characteristic equation.
Example 1: $y'' - 4y' + 13y = 3e^{2t}$, here $p(r) = r^2 - 4r + 13$ and $r_1 = 2 + 3i$, $r_2 = 2 - 3i$

Hence we use $Y(t) = A_0 e^{2t}$ and obtain $LY = A_0(4 - 4\cdot 2 + 13)e^{2t} \overset{!}{=} 3e^{2t}$
This gives $9A_0 = 3$, so $A_0 = \frac{1}{3}$ and $\boxed{Y(t) = \frac{1}{3}e^{2t}}$.

If μ **is a root of the characteristic equation with multiplicity** m use $\boxed{f(t) = A_0 t^m e^{\mu t}}$
Example 2: $y'' - 4y' + 4y = 3e^{2t}$, here $p(r) = r^2 - 4r + 4$ and $r_1 = r_2 = 2$
Hence we use $Y(t) = A_0 t^2 e^{2t}$. We use the **product rule** $(uv)' = u'v + uv'$, $(uv)'' = u''v + 2u'v' + uv''$
and obtain

$$LY = A_0\Big(\big(\boxed{2e^{2t}} + 8te^{2t} + t^2 4e^{2t}\big) - 4\big(2te^{2t} + t^2 2e^{2t}\big) + 4\big(t^2e^{2t}\big)\Big) = A_0 2e^{2t} \overset{!}{=} 3e^{2t} \tag{4}$$

Note that all the t^2e^{2t} and te^{2t} terms cancel out.
This is no coincidence: for a double root $r = 2$ we have $p(2) = 0$ and $p'(2) = 0$. Hence the **key identity** (3) gives

$$L(t^2e^{2t}) = \underbrace{p''(2)}_{2}e^{2t} + 2\underbrace{p'(2)}_{0}te^{2t} + \underbrace{p(2)}_{0}t^2e^{2t} = 2e^{2t}$$

since $p''(t) = 2$. We can use the key identities to simplify our work.
We obtain from (4) that $2A_0 = 3$ or $A_0 = \frac{3}{2}$ and $\boxed{Y(t) = \frac{3}{2}t^2e^{2t}}$.

Case 2: polynomial times exponential function $\boxed{f(t) = (\alpha_0 t^d + \alpha_1 t^{d-1} + \cdots + \alpha_d)e^{\mu t}}$

If μ **is NOT a root of the characteristic equation:** use $\boxed{Y(t) = (A_0 t^d + A_1 t^{d-1} + \cdots + A_d)e^{\mu t}}$.

If μ **is a root of the characteristic equation with multiplicity** m use $\boxed{Y(t) = t^m(A_0 t^d + A_1 t^{d-1} + \cdots + A_d)e^{\mu t}}$

Example 3: $y''-4y'+13y=te^{3t}$, here $p(r)=r^2-4r+13$ and $r_1=2+3i$, $r_2=2-3i$
Hence we use $Y(t)=(A_0t+A_1)e^{3t}$. We now have to find $LY=A_0L(te^{2t})+A_1L(e^{2t})$. Here we use the key identities (1), (2): With $p(3)=3^2-4\cdot 3+13=10$, $p'(r)=2r-4$, $p'(3)=2$ we get

$$Le^{3t}=p(3)e^{3t}=10e^{3t}, \qquad L(te^{3t})=p'(3)e^{3t}+p(3)te^{3t}=2e^{3t}+10te^{3t}$$
$$LY=(A_0(2+10t)+10A_1)\,e^{3t}\overset{!}{=}(1\cdot t+0)\,e^{3t}$$

Hence $10A_0=1$ and $2A_0+10A_1=0$. This gives $A_0=\frac{1}{10}$ and $A_1=-\frac{1}{50}$, so $\boxed{Y(t)=\left(\frac{1}{10}t-\frac{1}{50}\right)e^{3t}}$

Special case $\mu=0$: polynomial $\boxed{f(t)=\alpha_0t^d+\alpha_1t^{d-1}+\cdots+\alpha_d}$

If 0 is NOT a root of the characteristic equation: use $\boxed{Y(t)=A_0t^d+A_1t^{d-1}+\cdots+A_d}$.
If 0 is a root of the characteristic equation with multiplicity m use $\boxed{Y(t)=t^m(A_0t^d+A_1t^{d-1}+\cdots+A_d)}$

Case 3: exponential function times sine/cosine $\boxed{f(t)=\alpha_0\cos(\nu t)e^{\mu t}+\beta_0\sin(\nu t)e^{\mu t}}$

If $\mu+i\nu$ is NOT a root of the characteristic equation: use $\boxed{Y(t)=A_0\cos(\nu t)e^{\mu t}+B_0\sin(\nu t)e^{\mu t}}$
If $\mu+i\nu$ is a root of the characteristic equation with multiplicity m use $\boxed{Y(t)=t^m\left(A_0\cos(\nu t)e^{\mu t}+B_0\sin(\nu t)e^{\mu t}\right)}$
Example 4: $y''-4y'+4y=\sin t$. Recall $r_1=r_2=2$. Here we have $\mu=0$, $\nu=1$, so $\mu+i\nu=i$ is not a root of the characteristic equation.
Use $Y(t)=A_0\cos t+B_0\sin t$, then

$$LY=Y''-4Y'+4Y=A_0(-\cos t+4\sin t+4\cos t)+A_1(-\sin t-4\cos t+4\sin t)$$
$$=(3A_0-4A_1)\cos t+(4A_0+3A_1)\sin t\overset{!}{=}0\cdot\cos t+1\cdot\sin t$$

Hence $3A_0-4A_1=0$ and $4A_0+3A_1=1$. This linear system has the solution $A_0=\frac{4}{25}$, $A_1=\frac{3}{25}$, so $\boxed{Y(t)=(4\cos t+3\sin t)/25}$
Example 5: $y''-4y'+4y=e^{-t}\sin t$. Recall $r_1=r_2=2$. Here we have $\mu=-1$, $\nu=1$, so $\mu+i\nu=-1+i$ is not a root of the characteristic equation.
Here we need to use $Y(t)=A_0e^{-t}\cos t+A_1e^{-t}\sin t$. I skip the details, but this will give $A_0=\frac{3}{50}$, $B_0=\frac{2}{25}$.

General case (includes all of the above) $\boxed{f(t)=(\alpha_0t^d+\cdots+\alpha_d)\cos(\nu t)e^{\mu t}+(\beta_0t^d+\cdots+\beta_d)\sin(\nu t)e^{\mu t}}$

Check whether $\mu+i\nu$ is a root of the characteristic equation, let m denote the multiplicity ($m=0$ if $\mu+i\nu$ is not a root of the characteristic equation).

Use $\boxed{Y(t)=t^m\left((A_0t^d+\cdots+A_d)\cos(\nu t)e^{\mu t}+(B_0t^d+\cdots+B_d)\sin(\nu t)e^{\mu t}\right)}$

Note:

- this includes all previous cases:
  for $\mu=0$ we get $f(t)=(\alpha_0t^d+\cdots+\alpha_d)\cos(\nu t)+(\beta_0t^d+\cdots+\beta_d)\sin(\nu t)$
  for $\nu=0$ we get $f(t)=(\alpha_0t^d+\cdots+\alpha_d)e^{\mu t}$
  for $d=0$ we get $f(t)=\alpha_0\cos(\nu t)e^{\mu t}+\beta_0\sin(\nu t)e^{\mu t}$
- we **have to use all terms**, even if some coefficients in $f(t)$ are zero.
  Example 6: for $f(t)=t^2\sin(7t)$ we have to use $Y(t)=t^m\left((A_0t^2+A_1t+A_2)\cos(7t)+(B_0t^2+B_1+B_2)\sin(7t)\right)$ with six unknown coefficients.
- If $f(t)$ is a sum of several terms we can find the corresponding particular solutions separately, and then add them together.
  Example 7: $y''-2y'+y=t\sin t+\cos t+\cos(2t)$
  characteristic equation is $r^2-2r+1=0$, hence $r_1=1$, $r_2=-1$. This gives the fundamental set $Y_1(t)=e^t$, $Y_2(t)=e^{-t}$.
  We first consider $f_{(1)}(t)=t\sin t+\cos t$, let $Y_{(1)}(t)=(A_0t+A_1)\cos t+(B_0t+B_1)\sin t$ and find the constants A_0,A_1,B_0,B_1:

$$Y_{(1)}(t)=\left(\tfrac{1}{2}t+\tfrac{1}{2}\right)\cos t+(0\cdot t-1)\sin t$$

Then we consider $f_{(2)}(t) = \cos(2t)$, let $Y_{(2)}(t) = A_0' \cos(2t) + B_0' \sin(2t)$ and find the constants A_0', B_0'.

$$Y_{(2)}(t) = -\tfrac{3}{25}\cos(2t) - \tfrac{4}{25}\sin(2t)$$

Then the general solution of the ODE is

$$y(t) = c_1 e^t + c_2 e^{-t} + Y_{(1)}(t) + Y_{(2)}(t)$$

Alternative method for sine/cosine terms:

We can write $\boxed{f(t) = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t)}$ with $\alpha, \beta \in \mathbb{R}$ as the $\boxed{\text{real part of } ce^{(\mu+i\nu)t}}$ with complex $c = \alpha - i\beta$:

$$\operatorname{Re}\left[(\alpha - i\beta)e^{(\mu+i\nu)t}\right] = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t)$$

First assume that $\mu + i\nu$ is **not** a root of the characteristic polynomial $p(r)$.

- use the key identity: $Le^{rt} = p(r)e^{rt}$ with $r = \mu + i\nu$
- let $C = \dfrac{c}{p(\mu+i\nu)}$, then $L\left(Ce^{(\mu+i\nu)t}\right) = ce^{(\mu+i\nu)t}$ and taking the real part:

$$L\operatorname{Re}\left(Ce^{(\mu+i\nu)t}\right) = \operatorname{Re}\left(ce^{(\mu+i\nu)t}\right) = f$$

i.e., our particular solution is $\boxed{Y(t) = \operatorname{Re}\left(Ce^{(\mu+i\nu)t}\right) = e^{\mu t}\operatorname{Re}\left(\dfrac{(\alpha - i\beta)e^{i\nu t}}{p(\mu+i\nu)}\right)}$

This is the so-called **zero degree formula** if $\mu + i\nu$ is not a root of $p(r)$.

Example 8: $y'' - 2y' + y = 2\cos t + 3\sin t$. Here $\mu + i\nu = i$.

- $f(t) = 2\cos t + 3\sin t = \operatorname{Re}\left(ce^{(\mu+i\nu)t}\right)$ with $c = 2 - 3i$ and $\mu + i\nu = i$
- $p(r) = r^2 - 2r + 1$ and $p(i) = i^2 - 2i + 1 = -2i$
- find $C = \dfrac{c}{p(\mu+i\nu)} = \dfrac{2-3i}{-2i} = i + \frac{3}{2}$
- hence $\boxed{Y(t) = \operatorname{Re}\left[(i + \tfrac{3}{2})(\cos t + i\sin t)\right] = \tfrac{3}{2}\cos t - \sin t}$

Example 9: $y'' + y = 2\cos t + 3\sin t$. Here $p(r) = r^2 + 1$ and $\mu + i\nu = i$ which is a root of $p(r)$ with **multipicity** $m = 1$**.**

Hence we need to use $\boxed{te^{(\mu+i\nu)t}}$:

- use the key identity: $L\left(te^{rt}\right) = p'(r)e^{rt}$ with $r = \mu + i\nu$ (since $p(r) = 0$)
- here $p'(r) = 2r$ and $p'(i) = 2i$
- find $C = \dfrac{c}{p'(\mu+i\nu)} = \dfrac{2-3i}{2i} = -i - \frac{3}{2}$
- hence $\boxed{Y(t) = \operatorname{Re}\left(Cte^{(\mu+i\nu)t}\right) = \operatorname{Re}\left[(-i - \tfrac{3}{2})t(\cos t + i\sin t)\right] = t(-\tfrac{3}{2}\cos t + \sin t)}$
- here we get $\boxed{Y(t) = te^{\mu t}\operatorname{Re}\left(\dfrac{(\alpha - i\beta)e^{i\nu t}}{p'(\mu+i\nu)}\right)}$. This is the **zero degree formula for a root of multiplicity 1**.

If $\mu + i\nu$ is a **root of multiplicity** m we obtain the **zero degree formula** $\boxed{Y(t) = t^m e^{\mu t}\operatorname{Re}\left(\dfrac{(\alpha - i\beta)e^{i\nu t}}{p^{(m)}(\mu+i\nu)}\right)}$