Finding a particular solution using "variation of parameters"

We consider an **inhomogeneous linear ODE**.

We first consider the the case of **order 2**:

$$\underbrace{y'' + a_1(t)y' + a_0(t)y}_{Ly} = f(t)$$

Here the functions $a_1(t), a_0(t), f(t)$ are continuous for $t \in (a,b)$.

We want to find a function $Y(t)$ with $LY = f$.

We assume that **we have a fundamental set of solutions** $Y_1(t), Y_2(t)$. This means that the matrix

$$A(t) = \begin{bmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{bmatrix}$$

is nonsingular for all $t \in (a,b)$.

The general solution of the homogeneous ODE $Ly = 0$ is given by

$$y(t) = c_1 Y_1(t) + c_2 Y_2(t)$$

The key idea is to replace the constants c_1, c_2 with functions $u_1(t), u_2(t)$: We try to construct a particular solution $Y(t)$ of the form

$$\boxed{Y(t) = u_1(t)Y_1(t) + u_2(t)Y_2(t)} \tag{1}$$

We need to find two functions $u_1(t), u_2(t)$, so we need to specify **two conditions**. One condition is obviously $LY = f$. For the second condition we pick

$$u_1'Y_1 + u_2'Y_2 = 0$$

since this will make our life easier when taking derivatives:

$$Y' = u_1Y_1' + u_2Y_2' + \underbrace{(u_1'Y_1 + u_2'Y_2)}_{\stackrel{!}{=}0} \tag{2}$$

$$Y'' = u_1Y_1'' + u_2Y_2'' + (u_1'Y_1' + u_2'Y_2')$$

yielding

$$LY = u_1 \underbrace{LY_1}_{0} + u_2 \underbrace{LY_2}_{0} + (u_1'Y_1' + u_2'Y_2') \stackrel{!}{=} f$$

This means that we have to find functions $u_1(t), u_2(t)$ satisfying

$$\underbrace{\begin{bmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{bmatrix}}_{A(t)} \begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

Recall that the matrix $A(t)$ is nonsingular for all $t \in (a,b)$. Let $\vec{v}(t)$ denote the solution of the linear system $A(t)\vec{v}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Then

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \vec{v}(t) f(t)$$

Note that

$$\underbrace{\begin{bmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{bmatrix}}_{A} \begin{bmatrix} -Y_2 \\ Y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \underbrace{Y_1 Y_2' - Y_1' Y_2}_{\det A} \end{bmatrix}, \quad \text{hence } \boxed{\vec{v} = \frac{1}{\det A} \begin{bmatrix} -Y_2 \\ Y_1 \end{bmatrix}}$$

We obtain

$$u_j(t) = \int v_j(t) f(t) dt \qquad \text{(any antiderivative works)}$$
$$Y(t) = u_1(t) Y_1(t) + u_2(t) Y_2(t)$$

Example: $y'' + y = \dfrac{1}{\sin t}$ for $t \in (0, \pi)$
We have $p(r) = r^2 + 1$ with the roots $r_1 = i$, $r_2 = -i$, hence $Y_1(t) = \cos t$, $Y_2(t) = \sin(t)$ and

$$A(t) = \begin{bmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}, \qquad \det A(t) = \cos^2 t + \sin^2 t = 1, \qquad \vec{v}(t) = \frac{1}{\det A} \begin{bmatrix} -Y_2 \\ Y_1 \end{bmatrix} = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$

We obtain

$$u_1'(t) = -\sin t \cdot \frac{1}{\sin t} = -1, \qquad u_1(t) = -t$$
$$u_2'(t) = \cos t \cdot \frac{1}{\sin t}, \qquad u_2(t) = \ln(\sin t)$$
$$Y(t) = u_1(t) Y_1(t) + u_2(t) Y_2(t) = \boxed{-t \cdot \cos t + \sin t \cdot \ln(\sin t)}$$

Integrating from t_0 to t

Define

$$u_j(t) = \int_{t_0}^{t} v_j(s) f(s) dt, \qquad j = 1, 2$$

then $u_j(t_0) = 0$ and (1), (2) show that the particular solution $Y(t)$ satisfies the initial conditions

$$Y(t_0) = 0, \qquad Y'(t_0) = 0$$

Green function $G(t,s)$

Define the Green function

$$G(t,s) = Y_1(t) v_1(s) + Y_2(t) v_2(s)$$

then the particular solution is given by

$$\boxed{Y(t) = \int_{t_0}^{t} G(t,s) f(s) ds} \tag{3}$$

Consider a fixed value of s. Then the function $\tilde{y}(t) = G(t,s)$ is a solution of the homogeneous ODE $L\tilde{y} = 0$ satisfying the following **initial conditions at time** s

$$\begin{bmatrix} \tilde{y}(s) \\ \tilde{y}'(s) \end{bmatrix} = \underbrace{\begin{bmatrix} Y_1(s) & Y_2(s) \\ Y_1'(s) & Y_2'(s) \end{bmatrix}}_{A(s)} \begin{bmatrix} v_1(s) \\ v_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The last equality is the definition of $\vec{v}(s)$.

Case of constant coefficients

Assume $Ly = y'' + a_1 y' + a_0 y$ with constants a_0, a_1. Consider the initial value problem

$$Ly = 0, \qquad \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We solve the linear system $A(0)\vec{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and get the solution

$$g(t) = c_1 Y_1(t) + c_2 Y_2(t)$$

For a fixed value of s consider the function $\tilde{y}(t) = g(t-s)$. This solves the initial value problem

$$L\tilde{y} = 0, \qquad \begin{bmatrix} \tilde{y}(0) \\ \tilde{y}'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This is the same initial value problem which $G(t,s)$ satisfies. Hence we must have $\boxed{G(t,s) = g(t-s)}$ and the formula for the particular solution becomes

$$\boxed{Y(t) = \int_{t_0}^{t} g(t-s) f(s) ds}$$

"Variation of parameters" for higher order ODEs

Everything works for a linear ODE of arbitrary order. E.g., for a **linear ODE of order 3** we have

$$y''' + a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t)$$

We assume that **we have a fundamental set of solutions** $Y_1(t), Y_2(t), Y_3(t)$. This means that the matrix

$$A(t) = \begin{bmatrix} Y_1(t) & Y_2(t) & Y_3(t) \\ Y_1'(t) & Y_2'(t) & Y_3'(t) \\ Y_1''(t) & Y_2''(t) & Y_3''(t) \end{bmatrix}$$

is nonsingular for all $t \in (a,b)$.
The general solution of the homogeneous ODE $Ly = 0$ is given by

$$y(t) = c_1 Y_1(t) + c_2 Y_2(t) + c_3 Y_3(t)$$

The key idea is to replace the constants c_1, c_2, c_3 with functions $u_1(t), u_2(t), u_3(t)$: We try to construct a particular solution $Y(t)$ of the form

$$\boxed{Y(t) = u_1(t)Y_1(t) + u_2(t)Y_2(t) + u_3(t)Y_3(t)} \tag{4}$$

We need to find three functions $u_1(t), u_2(t), u_3(t)$, so we need to specify **three conditions**. One condition is obviously $LY = f$. For the other two conditions we pick

$$u_1'Y_1 + u_2'Y_2 + u_3'Y_3 = 0$$
$$u_1'Y_1' + u_2'Y_2' + u_3'Y_3' = 0$$

since this will make our life easier when taking derivatives:

$$Y' = u_1Y_1' + u_2Y_2' + u_3Y_3' + \underbrace{(u_1'Y_1 + u_2'Y_2 + u_3'Y_3)}_{\overset{!}{=} 0} \tag{5}$$

$$Y'' = u_1Y_1'' + u_2Y_2'' + u_3Y_3'' + \underbrace{(u_1'Y_1' + u_2'Y_2' + u_3'Y_3')}_{\overset{!}{=} 0} \tag{6}$$

$$Y''' = u_1Y_1''' + u_2Y_2''' + u_3Y_3''' + (u_1'Y_1'' + u_2'Y_2'' + u_3'Y_3'')$$

yielding

$$LY = u_1 \underbrace{LY_1}_{0} + u_2 \underbrace{LY_2}_{0} + u_3 \underbrace{LY_3}_{0} + (u_1'Y_1'' + u_2'Y_2'' + u_3Y_3'') \overset{!}{=} f$$

This means that we have to find functions $u_1(t), u_2(t), u_3(t)$ satisfying

$$\underbrace{\begin{bmatrix} Y_1(t) & Y_2(t) & Y_3(t) \\ Y_1'(t) & Y_2'(t) & Y_3'(t) \\ Y_1''(t) & Y_2''(t) & Y_3''(t) \end{bmatrix}}_{A(t)} \begin{bmatrix} u_1'(t) \\ u_2'(t) \\ u_3'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f(t) \end{bmatrix}$$

Recall that the matrix $A(t)$ is nonsingular for all $t \in (a,b)$. Let $\vec{v}(t)$ denote the solution of the linear system $A(t)\vec{v}(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Then

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \\ u_3'(t) \end{bmatrix} = \vec{v}(t) f(t)$$

We obtain

$$u_j(t) = \int v_j(t) f(t) dt$$
$$Y(t) = u_1(t)Y_1(t) + u_2(t)Y_2(t) + u_3(t)Y_3(t)$$

Integrating from t_0 to t

Define

$$u_j(t) = \int_{t_0}^{t} v_j(s) f(s) dt, \qquad j = 1,2,3$$

then $u_j(t_0) = 0$ and (4), (5), (6) give

$$Y(t_0) = 0, \qquad Y'(t_0) = 0, \qquad Y''(t_0) = 0$$

Green function $G(t,s)$

Define

$$G(t,s) = Y_1(t)v_1(s) + Y_2(t)v_2(s) + Y_3(t)v_3(s)$$

then

$$Y(t) = \int_{t_0}^{t} G(t,s) f(s) ds$$

Consider a fixed value of s. Then the function $\tilde{y}(t) = G(t,s)$ is a solution of the homogeneous ODE $L\tilde{y} = 0$ satisfying the following **initial conditions at time** s

$$\begin{bmatrix} \tilde{y}(s) \\ \tilde{y}'(s) \\ \tilde{y}''(s) \end{bmatrix} = \underbrace{\begin{bmatrix} Y_1(s) & Y_2(s) & Y_3(s) \\ Y_1'(s) & Y_2'(s) & Y_3'(s) \\ Y_1''(s) & Y_2''(s) & Y_3''(s) \end{bmatrix}}_{A(s)} \begin{bmatrix} v_1(s) \\ v_2(s) \\ v_3(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The last equality is the definition of $\vec{v}(s)$.

Case of constant coefficients

Assume $Ly = y'' + a_2 y'' + a_1 y' + a_0 y$ with constants a_0, a_1, a_2. Consider the initial value problem

$$Ly = 0, \qquad \begin{bmatrix} y(0) \\ y'(0) \\ y''(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We solve the linear system $A(0)\vec{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and get the solution

$$g(t) = c_1 Y_1(t) + c_2 Y_2(t) + c_3 Y_3(t)$$

For a fixed value of s consider the function $\tilde{y}(t) = g(t-s)$. This solves the initial value problem

$$L\tilde{y} = 0, \qquad \begin{bmatrix} \tilde{y}(0) \\ \tilde{y}'(0) \\ \tilde{y}''(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This is the same initial value problem which $G(t,s)$ satisfies. Hence we must have $\boxed{G(t,s) = g(t-s)}$ and the formula for the particular solution becomes

$$\boxed{Y(t) = \int_{t_0}^{t} g(t-s) f(s) ds}$$