"Variation of parameters" for linear system of ODEs

We consider a **linear ODE system**

$$\begin{bmatrix} y_1'(t) \\ \vdots \\ y_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$
$$\vec{y}'(t) = A(t)\vec{y}(t) + \vec{f}(t)$$

We assume that the functions $a_{jk}(t), f_j(t)$ are continuous for $t \in (\alpha, \beta)$.

We want to find a **particular solution** $\vec{Y}(t)$ satisfying $\vec{Y}'(t) = A(t)\vec{Y}(t) + \vec{f}(t)$.

We first consider the **homogeneous ODE** $\vec{y}'(t) = A(t)\vec{y}(t)$. We assume that **we have a fundamental set of solutions** $\vec{Y}^{(1)}(t), \dots, \vec{Y}^{(n)}(t)$. This means that the **fundamental matrix**

$$\Psi(t) = \left[\vec{Y}^{(1)}(t), \dots, \vec{Y}^{(n)}(t)\right]$$

is nonsingular for all $t \in (\alpha, \beta)$. Note that

$$\Psi'(t) = A(t)\Psi(t)$$

The general solution of the homogeneous ODE is given by

$$\vec{y}(t) = c_1\vec{Y}^{(1)}(t) + \dots + c_n\vec{Y}^{(n)}(t) = \Psi(t)\vec{c}$$

The key idea is to replace the constants $c_1, \dots, c_n$ with functions $u_1(t), \dots, u_n(t)$: We try to construct a particular solution $\vec{Y}(t)$ of the form

$$\boxed{\vec{Y}(t) = \Psi(t)\vec{u}(t)} \tag{1}$$

The function $\vec{Y}(t)$ should satisfy the ODE:

$$\begin{aligned} \vec{Y}'(t) - A(t)\vec{Y}(t) &= \vec{f}(t) \\ \Psi(t)\vec{u}'(t) + \underbrace{\Psi'(t)\vec{u}(t) - A(t)\Psi(t)\vec{u}(t)}_{0} &= \vec{f}(t) \end{aligned}$$

yielding the linear system

$$\boxed{\Psi(t)\vec{u}'(t) = \vec{f}(t)}$$

Recall that the matrix $\Psi(t)$ is nonsingular, so we can solve this linear system. We can write the solution using the inverse matrix $\Psi(t)^{-1}$:

$$\vec{u}'(t) = \Psi(t)^{-1}\vec{f}(t)$$

We obtain

$$\vec{u}(t) = \int \Psi(t)^{-1}\vec{f}(t)\,dt \qquad \text{(any antiderivative works)}$$
$$\vec{Y}(t) = \Psi(t)\vec{u}(t)$$

Integrating from t_0 to t

Define

$$\vec{u}(t) = \int_{t_0}^{t} \Psi(s)^{-1}\vec{f}(s)dt,$$

then $\vec{u}(t_0) = \vec{0}$ gives that $\vec{Y}(t) = \Psi(t)\vec{u}(t)$ satisfies the initial condition $\vec{Y}(t_0) = \vec{0}$.

Green function $G(t,s)$

We have

$$\vec{Y}(t) = \Psi(t)\vec{u}(t) = \int_{t_0}^{t} \Psi(t)\Psi(s)^{-1}\vec{f}(s)dt$$

Define the matrix valued function

$$\boxed{G(t,s) := \Psi(t)\Psi(s)^{-1}}$$

then the **particular solution** is

$$\boxed{\vec{Y}(t) = \int_{t_0}^{t} G(t,s)\vec{f}(s)ds} \tag{2}$$

Consider a fixed value of s. Then the matrix valued function $\Phi(t) := G(t,s)$ satisfies

$$\begin{aligned}\Phi'(t) &= A\Phi(t)\\ \Phi(s) &= \Psi(s)\Psi(s)^{-1} = I\end{aligned}$$

Therefore $\Phi(t)$ is a fundamental matrix satisfying $\Phi(s) = I$, i.e., $\Phi(t)$ is the "natural fundamental matrix" for $t_0 = s$.

Solution of initial value problem

Note that $\vec{Y}(t)$ solves the initial value problem with initial condition $\vec{Y}(0) = \vec{0}$
For the initial value problem

$$\vec{y}' = A(t)\vec{y}(t) + \vec{f}(t), \qquad \vec{y}(t_0) = \vec{y}^{(0)}$$

we can write the solution as

$$\boxed{\vec{y}(t) = G(t,t_0)\vec{y}^{(0)} + \int_{t_0}^{t} G(t,s)\vec{f}(s)ds}$$

- the first term satisfies the homogenous ODE and has initial value $\vec{y}^{(0)}$
- the second term satisfies the inhomogeneous ODE and has initial value $\vec{0}$

Case of constant coefficients

Assume that the matrix A has constant coefficients a_{jk}.
Let $\vec{e}^{(j)}$ denote the jth unit vector, i.e., $u_k^{(j)} = \begin{cases}1 & \text{for } j = k\\ 0 & \text{for } j \neq k\end{cases}$. We have $I = \left[\vec{e}^{(1)}, \ldots, \vec{e}^{(n)}\right]$.
Let $\vec{y}_{(j)}(t)$ denote the solution of the initial value problem

$$\vec{y}' = A\vec{y}, \qquad \vec{y}(0) = \vec{e}^{(j)}$$

Let $\mathcal{G}(t)$ denote the matrix valued function

$$\mathcal{G}(t) = \left[\vec{y}_{(1)}(t), \ldots, \vec{y}_{(n)}(t)\right]$$

This function satisfies

$$\begin{aligned}\mathcal{G}'(t) &= A\mathcal{G}(t)\\ \mathcal{G}(0) &= I\end{aligned}$$

i.e., this is the "natural fundamental matrix" for $t_0 = 0$.
For a fixed value of s consider the shifted function $\Phi(t) = \mathcal{G}(t-s)$. This function satisfies

$$\begin{aligned}\Phi'(t) &= A\Phi(t)\\ \Phi(s) &= I\end{aligned}$$

This is the same initial value problem which $G(t,s)$ satisfies. Hence we must have $\boxed{G(t,s) = \mathcal{G}(t-s)}$ and the formula for the **particular solution** becomes

$$\boxed{\vec{Y}(t) = \int_{t_0}^{t} \mathcal{G}(t-s)\vec{f}(s)ds}$$

The **solution of the initial value problem** $\vec{y}\,'(t) = A\vec{y}(t) + \vec{f}(t)$ with initial condition $\vec{y}(t_0) = \vec{y}^{(0)}$ is given by

$$\boxed{\vec{y}(t) = \mathcal{G}(t-t_0)\vec{y}^{(0)} + \int_{t_0}^{t} \mathcal{G}(t-s)\vec{f}(s)ds}$$