Final Exam Solutions: MATH 410
Thursday, 14 December 2017
Professor David Levermore

1. [10] Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Give negations of each of the following assertions.

   (a) For every $\epsilon > 0$ there exists an $n_\epsilon \in \mathbb{N}$ such that
$$m, n > n_\epsilon \implies |x_m - x_n| < \epsilon\,.$$

   (b) $\lim_{n\to\infty} x_n = \infty$.

   Solution (a). There exists an $\epsilon > 0$ such that for every $N \in \mathbb{N}$ there exists $m, n \in \mathbb{N}$ such that
$$m, n > N \quad \text{and} \quad |x_m - x_n| \geq \epsilon\,.$$

   Solution (b). There are several acceptable answers. The shortest is
$$\liminf_{n\to\infty} x_n < \infty\,.$$
   This could be expanded as
$$\exists M > 0 \quad \text{such that} \quad x_n \leq M \quad \text{frequently as } n \to \infty\,,$$
   which could be expanded further as
$$\exists M > 0 \quad \text{such that} \quad \forall m \in \mathbb{N} \quad \exists n > m \quad \text{such that} \quad x_n \leq M\,.$$
   The last two answers can also be obtained by first expressing $\lim_{n\to\infty} x_n = \infty$ either as
$$\forall M > 0 \quad x_n > M \quad \text{eventually as } n \to \infty\,,$$
   or as
$$\forall M > 0 \quad \exists m \in \mathbb{N} \quad \text{such that} \quad \forall n > m \quad x_n > M\,,$$
   and then simply negating. □

2. [15] Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be bounded, positive sequences in $\mathbb{R}$.

   (a) [10] Prove that
$$\limsup_{k\to\infty}\,(a_k b_k) \leq \Big(\limsup_{k\to\infty}\, a_k\Big)\Big(\limsup_{k\to\infty}\, b_k\Big)\,.$$

   (b) [5] Give an example for which equality does not hold above.

   Remark. This problem is from Exam 1.

   Solution (a). Let $c_k = a_k b_k$ for every $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ we define
$$\bar{a}_k = \sup\{a_l : l \geq k\}\,, \qquad \bar{b}_k = \sup\{b_l : l \geq k\}\,, \qquad \bar{c}_k = \sup\{c_l : l \geq k\}\,.$$
   Because the sequences $\{a_k\}_{k\in\mathbb{N}}$, $\{b_k\}_{k\in\mathbb{N}}$, and $\{c_k\}_{k\in\mathbb{N}}$ are bounded above and positive, for every $k \in \mathbb{N}$ we have
$$0 < \bar{a}_k < \infty\,, \qquad 0 < \bar{b}_k < \infty\,, \qquad 0 < \bar{c}_k < \infty\,.$$

The real sequences $\{\overline{a}_k\}_{k\in\mathbb{N}}$, $\{\overline{b}_k\}_{k\in\mathbb{N}}$, and $\{\overline{c}_k\}_{k\in\mathbb{N}}$ are nonincreasing because their terms are supremums of successively smaller sets. Moreover, they are bounded below because $\{a_k\}_{k\in\mathbb{N}}$, $\{b_k\}_{k\in\mathbb{N}}$, and $\{c_k\}_{k\in\mathbb{N}}$ are positive. Therefore they converge by Montonic Sequence Convergence Theorem. By the definition of lim sup we have

$$\limsup_{k\to\infty} a_k = \lim_{k\to\infty} \overline{a}_k \,, \qquad \limsup_{k\to\infty} b_k = \lim_{k\to\infty} \overline{b}_k \,, \qquad \limsup_{k\to\infty} c_k = \lim_{k\to\infty} \overline{c}_k \,.$$

The crucial observation is that for every $k \in \mathbb{N}$ we have

$$c_l = a_l b_l \leq \overline{a}_k \overline{b}_k \quad \text{for every } l \geq k \,,$$

which yields the inequality

$$\overline{c}_k = \sup\{c_l \,:\, l \geq k\} \leq \overline{a}_k \overline{b}_k \,.$$

This inequality and the properties of limits then imply

$$\begin{aligned} \limsup_{k\to\infty} a_k b_k = \limsup_{k\to\infty} c_k = \lim_{k\to\infty} \overline{c}_k \leq \lim_{k\to\infty} \overline{a}_k \overline{b}_k &= \Big(\lim_{k\to\infty} \overline{a}_k\Big)\Big(\lim_{k\to\infty} \overline{b}_k\Big) \\ &= \Big(\limsup_{k\to\infty} a_k\Big)\Big(\limsup_{k\to\infty} b_k\Big) \,. \end{aligned}$$

This is the inequality that we were asked to prove. □

Solution (b). Let $\rho > 1$. Let $\{a_k\}_{k\in\mathbb{N}}$ be any bounded, positive sequence such that

$$\liminf_{k\to\infty} a_k = \frac{1}{\rho} \,, \qquad \text{and} \qquad \limsup_{k\to\infty} a_k = \rho \,.$$

For example, we can simply take

$$a_k = \rho^{(-1)^k} = \begin{cases} \rho & \text{for } k \text{ even} \\ \dfrac{1}{\rho} & \text{for } k \text{ odd} \,. \end{cases}$$

Set $b_k = 1/a_k$ for every $k \in \mathbb{N}$. Then

$$\limsup_{k\to\infty} b_k = \limsup_{k\to\infty} \frac{1}{a_k} = \frac{1}{\liminf_{k\to\infty} a_k} = \rho \,,$$

whereby $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ are bounded, positive sequences in $\mathbb{R}$ such that

$$\limsup_{k\to\infty} a_k b_k = 1 < \rho^2 = \Big(\limsup_{k\to\infty} a_k\Big)\Big(\limsup_{k\to\infty} b_k\Big) \,.$$

So equality does not hold for the bounded, positive sequences $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$. □

3. [15] Let $f : (a, b) \to \mathbb{R}$ be differentiable at a point $c \in (a, b)$ with $f'(c) < 0$. Show that there exists a $\delta > 0$ such that

$$\begin{aligned} x \in (c-\delta, c) \subset (a, b) &\implies f(x) > f(c) \,, \\ x \in (c, c+\delta) \subset (a, b) &\implies f(c) > f(x) \,, \end{aligned}$$

Remark. This problem is from Exam 2. It asks you to prove the Transversality Lemma.

Solution. Because f is differentiable at c we know that

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \,.$$

By the ϵ-δ definition of limit, this means that for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x \in (a, b)$ we have

$$0 < |x - c| < \delta \qquad \Longrightarrow \qquad \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \epsilon \,.$$

Because $f'(c) < 0$ we may take $\epsilon = -f'(c)$ above to conclude that there exists $\delta > 0$ such that for every $x \in (a, b)$ we have

$$0 < |x - c| < \delta \qquad \Longrightarrow \qquad \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < -f'(c) \,.$$

Because $c \in (a, b)$ we may assume that δ is small enough so that $(c - \delta, c + \delta) \subset (a, b)$. Then we have

$$0 < |x - c| < \delta \qquad \Longrightarrow \qquad 2f'(c) < \frac{f(x) - f(c)}{x - c} < 0 \,.$$

This implies that $x - c$ and $f(x) - f(c)$ will have opposite signs when $0 < |x - c| < \delta$. It follows that

$$\begin{aligned} x \in (c - \delta, c) \quad &\Longrightarrow \quad x - c < 0 \quad \Longrightarrow \quad f(x) - f(c) > 0 \,, \\ x \in (c, c + \delta) \quad &\Longrightarrow \quad x - c > 0 \quad \Longrightarrow \quad f(x) - f(c) < 0 \,. \end{aligned}$$

Because $(c - \delta, c) \subset (a, b)$ and $(c, c + \delta) \subset (a, b)$, the result follows. □

4. [20] Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be Riemann integrable over $[a, b]$. Prove that $f + g$ is Riemann integrable over $[a, b]$.

Remark. This problem is from Homework 12.

Solution. Let D_f, D_g, and D_{f+g} denote the points in $[a, b]$ at which f, g, and $f + g$ respectively are discontinuous. It is clear that $D_{f+g} \subset D_f \cup D_g$ because $f + g$ is continuous at every point where both f and g are continuous. Because f and g are Riemann integrable over $[a, b]$, one direction of the Lebesgue Theorem implies that D_f and D_g have measure zero. Because the union of two measure zero sets also has measure zero, and because any subset of a measure zero set also has measure zero, we see that $D_{f+g} \subset D_f \cup D_g$ has measure zero. The other direction of the Lebesgue Theorem then implies that $f + g$ is Riemann integrable over $[a, b]$. □

Alternative Solution. Let $\epsilon > 0$. Because f and g are Riemann integrable over $[a.b]$, the Darboux Theorem implies that there exist partitions P_ϵ^f and P_ϵ^g of $[a.b]$ such that

$$0 \le U(f, P_\epsilon^f) - L(f, P_\epsilon^f) < \frac{\epsilon}{2} \,, \qquad 0 \le U(g, P_\epsilon^g) - L(f, P_\epsilon^g) < \frac{\epsilon}{2} \,.$$

Set $P_\epsilon = P_\epsilon^f \vee P_\epsilon^g$. Then

$$\begin{aligned} U(f + g, P_\epsilon) &\le U(f, P_\epsilon) + U(g, P_\epsilon) \le U(f, P_\epsilon^f) + U(g, P_\epsilon^g) \,, \\ L(f + g, P_\epsilon) &\ge L(f, P_\epsilon) + L(g, P_\epsilon) \ge L(f, P_\epsilon^f) + L(g, P_\epsilon^g) \,. \end{aligned}$$

Upon combining the above inequalities we find that

$$\begin{aligned} 0 &\le U(f + g, P_\epsilon) - L(f + g, P_\epsilon) \\ &\le U(f, P_\epsilon^f) - L(f, P_\epsilon^f) + U(g, P_\epsilon^g) - L(g, P_\epsilon^g) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \,. \end{aligned}$$

Because such a P_ϵ can be found for every $\epsilon > 0$, the Darboux Theorem implies that $f + g$ is Riemann integrable. □

Remark. The second solution gives a lot more. It is just a few steps away from showing that the intergral of $f + g$ is the sum of the integrals of f and g.

5. [25] Consider a function g defined by

$$g(x) = \sum_{k=0}^{\infty} \frac{1}{3^k} \sin(2^k x)\,,$$

for every $x \in \mathbb{R}$ for which the above series converges.

(a) [10] Show that g is defined for every $x \in \mathbb{R}$.

(b) [15] Show that g is continuously differentiable over $\mathbb{R}$ and that

$$g'(x) = \sum_{k=0}^{\infty} \frac{2^k}{3^k} \cos(2^k x)\,.$$

Remark. This problem is similar to one from Homework 15.

Solution (a). Because

$$\sum_{k=0}^{\infty} \frac{1}{3^k} \qquad \text{is a } \textit{geometric series} \text{ with } r = \tfrac{1}{3} < 1\,,$$

it is convergent. Because for every $k \in \mathbb{N}$ we have the bound

$$\left| \frac{1}{3^k} \sin(2^k x) \right| \leq \frac{1}{3^k} \qquad \text{for every } x \in \mathbb{R}\,,$$

the absolute comparison test says that the series defining $g(x)$ converges absolutely for every $x \in \mathbb{R}$. □

Solution (b). For every $n \in \mathbb{N}$ define the function g_n by

$$g_n(x) = \sum_{k=0}^{n} \frac{1}{3^k} \sin(2^k x) \qquad \text{for every } x \in \mathbb{R}\,.$$

Then g_n is continuously differentiable with

$$g_n'(x) = \sum_{k=0}^{n} \frac{2^k}{3^k} \cos(2^k x) \qquad \text{for every } x \in \mathbb{R}\,.$$

For every $k \in \mathbb{N}$ we have the bounds

$$\left\| \frac{1}{3^k} \sin(2^k x) \right\|_{\mathrm{B}(\mathbb{R})} \leq \frac{1}{3^k}\,, \qquad \left\| \frac{2^k}{3^k} \cos(2^k x) \right\|_{\mathrm{B}(\mathbb{R})} \leq \frac{2^k}{3^k}\,.$$

Then because the *geometric series*

$$\sum_{k=0}^{\infty} \frac{1}{3^k}\,, \qquad \sum_{k=0}^{\infty} \frac{2^k}{3^k}\,,$$

are convergent, the Weierstrass M-Test implies that the sequences of functions $\{g_n\}_{n\in\mathbb{N}}$ and $\{g_n'\}_{n\in\mathbb{N}}$ converge uniformly. Therefore $g_n \to g$ uniformly over $\mathbb{R}$.

Let $g_n' \to h$ uniformly over $\mathbb{R}$. Then h is continuous and g is continuously differentiable with $g' = h$. Therefore

$$g'(x) = h(x) = \sum_{k=0}^{\infty} \frac{2^k}{3^k} \cos(2^k x)\,.$$

□

6. [25] For every $n \in \mathbb{Z}_+$ define $h_n(x) = nx(1+nx)^{-2}$ for every $x \in [0,\infty)$.
   (a) [5] Prove that $h_n \to 0$ pointwise over $[0,\infty)$.
   (b) [10] Prove that this limit is not uniform over $[0,\infty)$.
   (c) [10] Prove that this limit is uniform over $[\delta,\infty)$ for every $\delta > 0$.

Solution (a). Because $h_n(0) = 0$ for every $n \in \mathbb{N}$, the convergence of $\{h_n(x)\}$ when $x = 0$ is obvious.

Now let $x \in (0,\infty)$ and consider the sequence $\{h_n(x)\}$. Then for every $n \in \mathbb{Z}_+$ we have

$$0 < h_n(x) = \frac{nx}{(1+nx)^2} < \frac{1}{nx}\,.$$

Let $\epsilon > 0$. Pick $n_\epsilon \in \mathbb{N}$ such that $n_\epsilon > 1/(x\epsilon)$. Then for every $n \geq n_\epsilon$ we have

$$0 < h_n(x) < \frac{1}{nx} \leq \frac{1}{n_\epsilon x} < \epsilon\,.$$

But this implies that $\{h_n(x)\}$ converges to zero as $n \to \infty$.

Therefore $h_n \to 0$ pointwise over $[0,\infty)$. □

Solution (b). We must show that there exists $\epsilon > 0$ such that for every $m \in \mathbb{N}$ there exists $n > m$ and $x \in [0,\infty)$ such that $h_n(x) \geq \epsilon$. This is easy to do. In fact, for every $n \in \mathbb{Z}_+$ we have $h_n(\frac{1}{n}) = \frac{1}{4}$. Therefore any $\epsilon \in (0, \frac{1}{4})$ works. □

Solution (c). Let $\delta > 0$. Then for every $n \in \mathbb{Z}_+$ and every $x \in [\delta,\infty)$ we have

$$0 < h_n(x) = \frac{nx}{(1+nx)^2} < \frac{1}{nx} \leq \frac{1}{n\delta}\,.$$

Let $\epsilon > 0$. Pick $n_\epsilon \in \mathbb{N}$ such that $n_\epsilon > 1/(\delta\epsilon)$. Then for every $n \geq n_\epsilon$ and every $x \in [\delta,\infty)$ we have

$$0 < h_n(x) < \frac{1}{n\delta} \leq \frac{1}{n_\epsilon \delta} < \epsilon\,.$$

Therefore $h_n \to 0$ uniformly over $[\delta,\infty)$. □

7. [20] Determine the set of $a \in \mathbb{R}$ for which the following formal infinite series converge. Give your reasoning.
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{a^n}{n3^n}$
   (b) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k^2+1}{k^4+1}\right)^a$

Remark. Part (b) of this problem is from Exam 1.

Solution (a). The series converges for every $a \in [-3, 3)$ and diverges otherwise.

The cases $|a| < 3$ and $|a| > 3$ are best handled by the Ratio Test. Let $b_n = a^n/(n3^n)$. Because

$$\lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} = \lim_{n\to\infty} \frac{n}{n+1}\,\frac{|a|}{3} = \frac{|a|}{3}\,,$$

the Ratio Test then implies that this series converges absolutely for $|a| < 3$ and diverges for $|a| > 3$.

The case $a = -3$ is best handled by the Alternating Series Test. Indeed, because the sequence

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \quad \text{is decreasing and positive}\,.$$

and because

$$\lim_{n\to\infty} \frac{1}{n} = 0\,,$$

the Alternating Series Test shows that

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \quad \text{converges}\,.$$

The case $a = 3$ reduces to the Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n}\,, \quad \text{which diverges}\,.$$

□

Solution (b). The series converges for $a \in (\frac{1}{2}, \infty)$ and diverges otherwise. Because

$$\frac{k^2+1}{k^4+1} \sim \frac{1}{k^2} \quad \text{as } k \to \infty\,,$$

we see that the original series should be compared with the p-series

$$\sum_{k=1}^{\infty} \frac{1}{k^{2a}}\,.$$

This is best handled by Two-Way Limit Comparison Test. Indeed, for every $a \in \mathbb{R}$ we have

$$\lim_{k\to\infty} \frac{\left(\dfrac{k^2+1}{k^4+1}\right)^a}{\dfrac{1}{k^{2a}}} = \lim_{k\to\infty} \left(\frac{k^4+k^2}{k^4+1}\right)^a = \lim_{k\to\infty} \left(\frac{1+\dfrac{1}{k^2}}{1+\dfrac{1}{k^4}}\right)^a = 1\,,$$

so the Two-Way Limit Comparison Test implies that

$$\sum_{k=0}^{\infty} \left(\frac{k^2+1}{k^4+1}\right)^a \quad \text{converges} \quad \Longleftrightarrow \quad \sum_{k=1}^{\infty} \frac{1}{k^{2a}} \quad \text{converges}\,.$$

Because the $p = 2a$ for the p-series, it converges for $a \in (\frac{1}{2}, \infty)$ and diverges otherwise. The same is thereby true for the original series. □

8. [20] Let $\alpha \in (0,1]$ and $K \in \mathbb{R}_+$ such that the function $f : [a,b] \to \mathbb{R}$ satisfy the Hölder bound

$$|f(x) - f(y)| < K\,|x-y|^\alpha \qquad \text{for every } x, y \in [a,b]\,.$$

(a) Show that f is uniformly continuous over $[a,b]$.

(b) Show that for every partition P of $[a,b]$ one has

$$0 \le U(f,P) - L(f,P) < |P|^\alpha K\,(b-a)\,.$$

Solution (a). Let $\epsilon > 0$. Set $\delta = (\epsilon/K)^{\frac{1}{\alpha}}$. Then for every $x, y \in [a,b]$ we have

$$|x-y| < \delta \quad \implies \quad |f(x) - f(y)| < K\,|x-y|^\alpha < K\,\delta^\alpha = \epsilon\,.$$

Hence, f is uniformly continuous over $[a,b]$. □

Solution (b). Let $P = [x_0, x_1, \cdots, x_n]$ be any partition of $[a,b]$. Then

$$0 \le U(f,P) - L(f,P) = \sum_{k=1}^{n} \big(\overline{m}_k - \underline{m}_k\big)(x_k - x_{k-1})\,,$$

where

$$\overline{m}_k = \sup\{f(x) \,:\, x \in [x_{k-1}, x_k]\}\,, \qquad \underline{m}_k = \inf\{f(x) \,:\, x \in [x_{k-1}, x_k]\}\,.$$

Because f is continuous over each $[x_{k-1}, x_k]$, the Extreme-Value Theorem implies that there exists points $\overline{x}_k$, $\underline{x}_k \in [x_{k-1}, x_k]$ such that

$$\overline{m}_k = f(\overline{x}_k) \qquad \text{and} \qquad \underline{m}_k = f(\underline{x}_k)\,.$$

The Hölder continuity of f then gives

$$\begin{aligned} 0 \le U(f,P) - L(f,P) &= \sum_{k=1}^{n} \big(f(\overline{x}_k) - f(\underline{x}_k)\big)(x_k - x_{k-1}) \\ &\le K \sum_{k=1}^{n} |\overline{x}_k - \underline{x}_k|^\alpha (x_k - x_{k-1})\,. \end{aligned}$$

Because $\overline{x}_k$, $\underline{x}_k \in [x_{k-1}, x_k]$ we have

$$|\overline{x}_k - \underline{x}_k| \le x_k - x_{k-1} \le \max\{x_m - x_{m-1} \,:\, m = 1, \cdots, n\} \equiv |P|\,,$$

whereby

$$\begin{aligned} 0 \le U(f,P) - L(f,P) &\le K \sum_{k=1}^{n} |P|^\alpha (x_k - x_{k-1}) \\ &= K\,|P|^\alpha \sum_{k=1}^{n} (x_k - x_{k-1}) = K\,|P|^\alpha\,(b-a)\,. \end{aligned}$$

□

9. [25] Given the fact that for every $x > -1$ and every $n \in \mathbb{Z}_+$ we have

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \log(1+x) = (-1)^{n-1} \frac{(n-1)!}{(1+x)^n},$$

prove that

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k \quad \text{for every } x \in (-1, 1],$$

and that the series diverges for every real $x \notin (-1, 1]$.

Remark. This problem is similar to one in Homework 14.

Partial Solution. It is easy to show that the series converges for every $x \in (-1, 1]$ and diverges for every real $x \notin (-1, 1]$.

Let $b_k = (-1)^{k-1} \frac{1}{k} x^k$. Because

$$\lim_{k\to\infty} \frac{|b_{k+1}|}{|b_k|} = \lim_{k\to\infty} \frac{k}{k+1} |x| = |x|,$$

the Ratio Test shows that the series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$. When $x = 1$ the Alternating Series Test shows that the series converges. When $x = -1$ the series is proportional to the Harmonic Series, so it must diverge. However, this is not a complete solution to the problem because these arguments do not show that when the series converges, it converges to $\log(1 + x)$.

Solution. Let $f(x) = \log(1+x)$ for every $x > -1$. We are given that for every $x > -1$ and every $n \in \mathbb{Z}_+$

$$f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{(1+x)^n},$$

whereby

$$f^{(n)}(0) = (-1)^{n-1}(n-1)! \quad \text{for every } n \in \mathbb{Z}_+.$$

Therefore the n^{th}-order Taylor approximation of f about 0 is

$$\mathcal{T}_0^n f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(0)\, x^k = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} x^k,$$

which is the n^{th} partial sum of the series. We must show that the associated Taylor remainder,

$$\mathcal{R}_0^n f(x) = f(x) - \mathcal{T}_0^n f(x),$$

vanishes as $n \to \infty$ for every $x \in (-1, 1]$.

The Cauchy Remainder Theorem states that

$$\begin{aligned}\mathcal{R}_0^n f(x) &= \frac{1}{n!} \int_0^x f^{(n+1)}(t)(x-t)^n \, \mathrm{d}t \\ &= (-1)^n \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}} \, \mathrm{d}t = (-1)^n \int_0^x \left(\frac{x-t}{1+t}\right)^n \frac{\mathrm{d}t}{1+t}.\end{aligned}$$

Consider the substitution

$$s = \frac{x-t}{1+t} = \frac{1+x}{1+t} - 1, \qquad t + 1 = \frac{1+x}{1+s}, \qquad \frac{\mathrm{d}t}{1+t} = -\frac{\mathrm{d}s}{1+s}.$$

Notice that s goes monotonically from x to 0 as t goes monotonically from 0 to x. This substitution yields

$$\mathcal{R}_0^n f(x) = (-1)^n \int_0^x \frac{s^n}{1+s}\, \mathrm{d}s\,.$$

If $x \geq 0$ then we have the bound

$$\left|\mathcal{R}_0^n f(x)\right| = \int_0^x \frac{s^n}{1+s}\, \mathrm{d}s \leq \int_0^x s^n\, \mathrm{d}s = \frac{x^{n+1}}{n+1}\,.$$

This bound vanishes as $n \to \infty$ for every $x \in [0,1]$. If $x < 0$ then we have the bound

$$\left|\mathcal{R}_0^n f(x)\right| = \int_0^{|x|} \frac{s^n}{1-s}\, \mathrm{d}s < \frac{1}{1-|x|} \int_0^{|x|} s^n\, \mathrm{d}s = \frac{1}{1-|x|}\,\frac{|x|^{n+1}}{n+1}\,.$$

This bound vanishes as $n \to \infty$ for every $x \in (-1,0)$. Therefore for every $x \in (-1,1]$ we have

$$\left|\mathcal{R}_0^n f(x)\right| \to 0 \quad \text{as } n \to \infty\,.$$

Collecting all of our results, we have shown that

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\, x^k \quad \text{for every } x \in (-1,1]\,,$$

and that the series diverges for all real $x \notin (-1,1]$. □

10. [25] For every $n \in \mathbb{Z}_+$ define $f_n(x) = n(1+nx)^{-2}$ for every $x \in [0,\infty)$.
   (a) [10] Prove for every $\delta > 0$ that
   $$\lim_{n\to\infty} f_n = 0 \qquad \text{uniformly over } [\delta,\infty)\,.$$
   (b) [5] Prove for every $\delta > 0$ that
   $$\lim_{n\to\infty} \int_0^\delta f_n = 1\,.$$
   (c) [10] Let $g : [0,1] \to \mathbb{R}$ be continuous. Show that
   $$\lim_{n\to\infty} \int_0^1 f_n g = g(0)\,.$$

Solution (a). Let $\delta > 0$. Because $f_n(x) = n(1+nx)^{-2}$ is a decreasing function of x over $[0,\infty)$, for every $x \geq \delta$ we have

$$|f_n(x)| = \left|\frac{n}{(1+nx)^2}\right| \leq \left|\frac{n}{(1+n\delta)^2}\right| < \frac{1}{n\,\delta^2}\,.$$

Let $\epsilon > 0$. Pick $n_\epsilon \in \mathbb{N}$ such that $n_\epsilon > 1/(\delta^2\epsilon)$. Then for every $n \geq n_\epsilon$ we have

$$|f_n(x)| < \frac{1}{n\,\delta^2} \leq \frac{1}{n_\epsilon\,\delta^2} < \epsilon \quad \text{for every } x \in [\delta,\infty)\,.$$

Therefore $f_n \to 0$ uniformly over $[\delta,\infty)$. □

Solution (b). Let $\delta > 0$. By the First Fundamental Theorem of Calculus we have

$$\int_0^\delta f_n = \int_0^\delta \frac{n}{(1+nx)^2}\,\mathrm{d}x = -\frac{1}{1+nx}\bigg|_0^\delta = 1 - \frac{1}{1+n\delta}\,.$$

Therefore

$$\lim_{n\to\infty}\int_0^\delta f_n = \lim_{n\to\infty}\left(1 - \frac{1}{1+n\delta}\right) = 1\,.$$

Solution (c). Assertion (b) implies that

$$\lim_{n\to\infty}\int_0^1 f_n\,\mathrm{d}x = 1\,,$$

whereby assertion (c) is equivalent to

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,\big(g(x)-g(0)\big)\,\mathrm{d}x = 0\,.$$

But this will follow if we can show that for every $\epsilon > 0$ we have

$$\limsup_{n\to\infty}\int_0^1 f_n(x)\,\big|g(x)-g(0)\big|\,\mathrm{d}x \leq \epsilon\,.$$

Let $\epsilon > 0$. Because g is continous at 0, there exists $\delta > 0$ such that

$$x \in [0,\delta) \qquad \Longrightarrow \qquad \big|g(x)-g(0)\big| < \epsilon\,.$$

Because g is continuous over the closed, bounded set $[0,1]$, the Extreme-Value Theorem implies g is bounded over $[0,1]$. Let $M = \sup\{|g(x)| \,:\, x \in [0,1]\}$, so $|g(x)-g(0)| \leq 2M$ for every $x \in [0,1]$. Then for every $n \in \mathbb{N}$

$$\begin{aligned}
\int_0^1 f_n(x)\big|g(x)-g(0)\big|\,\mathrm{d}x &= \int_0^\delta f_n(x)\big|g(x)-g(0)\big|\,\mathrm{d}x + \int_\delta^1 f_n(x)\big|g(x)-g(0)\big|\,\mathrm{d}x \\
&\leq \int_0^\delta f_n(x)\,\epsilon\,\mathrm{d}x + \int_\delta^1 f_n(x)\,2M\,\mathrm{d}x \\
&\leq \epsilon\int_0^\delta f_n + 2M\int_\delta^1 f_n\,.
\end{aligned}$$

Assertion (b) and the uniform convergence of assertion (a) imply that

$$\lim_{n\to\infty}\int_0^\delta f_n = 1\,, \qquad \lim_{n\to\infty}\int_\delta^1 f_n = 0\,,$$

whereby the previous inequality shows that

$$\limsup_{n\to\infty}\int_0^1 f_n(x)\big|g(x)-g(0)\big|\,\mathrm{d}x \leq \epsilon\,\lim_{n\to\infty}\int_0^\delta f_n + 2M\,\lim_{n\to\infty}\int_\delta^1 f_n = \epsilon\,.$$

But as was argued above, because this holds for every $\epsilon > 0$, assertion (c) follows. □

Remark. This problem is similar to a problem in Homework 14 and a problem in Homework 15.