Second In-Class Exam
Math 410, Professor David Levermore
Thursday, 2 November 2017

No books, notes, calculators, or any electronic devices. Indicate your answer to each part of each question clearly. Cross out any work that you do not want to be considered. **Your reasoning must be given for full credit.** Good luck!

1. [10] Give (with reasoning) a counterexample to each of the following false assertions.
   (a) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and increasing over $\mathbb{R}$ then $f' > 0$ over $\mathbb{R}$.
   (b) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable then its derivative $f' : \mathbb{R} \to \mathbb{R}$ is continuous.

2. [10] Let $f : (a, b) \to \mathbb{R}$ be differentiable at a point $c \in (a, b)$ with $f'(c) < 0$. Show that there exists a $\delta > 0$ such that
$$x \in (c - \delta, c) \subset (a, b) \implies f(x) > f(c)\,,$$
$$x \in (c, c + \delta) \subset (a, b) \implies f(c) > f(x)\,,$$

3. [10] Evaluate the following limit. (You may use theorems from class.)
$$\lim_{x\to 3} \frac{x^4 - 81}{x^3 - 27}\,.$$

4. [15] If $f(x) = \sin(x)$ for every $x \in \mathbb{R}$ then for every $k \in \mathbb{N}$ we have
$$f^{(2k)}(x) = (-1)^k \sin(x)\,, \qquad f^{(2k+1)}(x) = (-1)^k \cos(x) \qquad \text{for every } x \in \mathbb{R}\,.$$
Use this fact to show that
$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1} \quad \text{for every } x \in \mathbb{R}\,.$$

5. [10] Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be uniformly continuous over D. Let $\{x_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence contained in D. Show that $\{f(x_k)\}_{k\in\mathbb{N}}$ is a convergent sequence.

6. [10] Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Let c be a limit point of D. Write negations of the following assertions.
   (a) "For every sequence $\{x_k\}_{k\in\mathbb{N}} \subset D - \{c\}$ we have
$$\lim_{k\to\infty} |x_k - c| = 0 \quad \implies \quad \lim_{k\to\infty} f(x_k) = -\infty\,.\text{"}$$
   (b) "For every $M \in \mathbb{R}$ there exists a $\delta > 0$ such that for every $x \in D$ we have
$$0 < |x - c| < \delta \quad \implies \quad f(x) > M\,.\text{"}$$

7. [15] Prove that for every nonzero $x \in \mathbb{R}$ we have
$$1 + \tfrac{8}{7}x < (1 + x)^{\frac{8}{7}}\,.$$

8. [10] Show that the function $f(x) = 1/x$ is not uniformly continuous over $\mathbb{R}_+$.

9. [10] Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Suppose the equation $f'(x) = 0$ has at most five real solutions. Prove that the equation $f(x) = 0$ has at most six real solutions.