Second In-Class Exam
Math 410, Professor David Levermore
Thursday, 5 November 2015

1. [10] Give a counterexample to each of the following false assertions.
   (a) If $f : \mathbb{R} \to \mathbb{R}$ is increasing and one-to-one then it is also continuous.
   (b) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable then its derivative $f' : \mathbb{R} \to \mathbb{R}$ is continuous.

2. [10] Let $f : (a,b) \to \mathbb{R}$ be differentiable at a point $c \in (a,b)$ with $f'(c) > 0$. Prove that there exists a $\delta > 0$ such that
$$x \in (c-\delta, c) \subset (a,b) \quad \Longrightarrow \quad f(x) < f(c)\,,$$
$$x \in (c, c+\delta) \subset (a,b) \quad \Longrightarrow \quad f(c) < f(x)\,.$$

3. [10] Evaluate the following limit. (You may use theorems from class.)
$$\lim_{x\to 2} \frac{x^3 - 8}{x^2 - 4}\,.$$

4. [15] If $f(x) = \cosh(x) \equiv \frac{1}{2}(e^x + e^{-x})$ for every $x \in \mathbb{R}$ then for every $k \in \mathbb{N}$ we have
$$f^{(2k)}(x) = \cosh(x)\,, \qquad f^{(2k+1)}(x) = \sinh(x) \qquad \text{for every } x \in \mathbb{R}\,.$$
Use this fact to show that
$$\cosh(x) = \sum_{k=0}^{\infty} \frac{1}{(2k)!}\, x^{2k} \quad \text{for every } x \in \mathbb{R}\,.$$

5. [10] Let $D \subset \mathbb{R}$. A function $f : D \to \mathbb{R}$ is said to be Hölder continuous of order $\alpha \in (0,1]$ if there exists a $C > 0$ such that f satisfies the Hölder bound
$$|f(x) - f(y)| \leq C\,|x-y|^\alpha \qquad \text{for every } x, y \in D\,.$$
Prove that every such function is uniformly continuous over D.

6. [15] Prove that for every $x \in \mathbb{R}$ we have
$$1 + \tfrac{4}{3}x \leq (1+x)^{\frac{4}{3}}\,.$$

7. [10] Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Let c be a limit point of D. Write negations of the following assertions.
   (a) "For every sequence $\{x_k\}_{k\in\mathbb{N}} \subset D - \{c\}$ we have
$$\lim_{k\to\infty} |x_k - c| = 0 \quad \Longrightarrow \quad \lim_{k\to\infty} f(x_k) = \infty\,."$$
   (b) "For every $M \in \mathbb{R}$ there exists a $\delta > 0$ such that for every $x \in D$ we have
$$0 < |x - c| < \delta \quad \Longrightarrow \quad f(x) > M\,."$$

8. [10] Show that the function $f(x) = x^2$ is not uniformly continuous over $\mathbb{R}$.

9. [10] Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Suppose the equation $f'(x) = 0$ has at most one real solution. Prove that the equation $f(x) = 0$ has at most two real solutions.