Second In-Class Exam Solutions
Math 410, Professor David Levermore
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1. [10] Give a counterexample to each of the following false assertions.

(a) If $f : \mathbb{R} \to \mathbb{R}$ is increasing and one-to-one then it is also continuous.

Solution. There are many counterexamples. The simplest ones have a single jump discontinuity somewhere. For example, consider the function f defined by

$$f(x) = \begin{cases} x & \text{for } x < 0\,, \\ x + 1 & \text{for } x \geq 0\,. \end{cases}$$

This function is clearly increasing and one-to-one, but has a jump discontinuity at $x = 0$ because

$$\lim_{x \to 0^-} f(x) = 0 \neq 1 = \lim_{x \to 0^+} f(x)\,.$$

$\square$

(b) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable then its derivative $f' : \mathbb{R} \to \mathbb{R}$ is continuous.

Solution. There are many counterexamples. The one we discussed in class was

$$f(x) = \begin{cases} x^2 \cos\left(\dfrac{1}{x}\right) & \text{for } x \neq 0\,, \\ 0 & \text{for } x = 0\,. \end{cases}$$

This function is clearly continuously differentiable at every $x \neq 0$ with

$$f'(x) = 2x \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right) \quad \text{for } x \neq 0\,.$$

Moreover, it is differentiable at $x = 0$ with

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} x \cos\left(\frac{1}{x}\right) = 0\,.$$

Hence, f is differentiable over $\mathbb{R}$. However, because

$$\lim_{x \to 0} 2x \cos\left(\frac{1}{x}\right) = 0\,,$$

while

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) \quad \text{does not exist}\,,$$

it follows from the formula for $f'(x)$ given earlier that

$$\lim_{x \to 0} f'(x) \quad \text{does not exist}\,.$$

Therefore f' is not continuous at $x = 0$. $\square$

2. [10] Let $f : (a, b) \to \mathbb{R}$ be differentiable at a point $c \in (a, b)$ with $f'(c) > 0$. Prove that there exists a $\delta > 0$ such that

$$\begin{aligned} x \in (c - \delta, c) \subset (a, b) \quad &\Longrightarrow \quad f(x) < f(c)\,, \\ x \in (c, c + \delta) \subset (a, b) \quad &\Longrightarrow \quad f(c) < f(x)\,. \end{aligned}$$

Remark. The problem is asking you to prove the Transversality Lemma.

Solution. Because f is differentiable at c we know that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}\,.$$

By the ϵ-δ definition of limit, this means that for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x \in (a, b)$ we have

$$0 < |x - c| < \delta \quad \Longrightarrow \quad \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \epsilon\,.$$

Because $f'(c) > 0$ we may take $\epsilon = f'(c)$ above to conclude that there exists $\delta > 0$ such that for every $x \in (a, b)$ we have

$$0 < |x - c| < \delta \quad \Longrightarrow \quad \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < f'(c)\,.$$

Because $c \in (a, b)$ we may assume that δ is small enough so that $(c - \delta, c + \delta) \subset (a, b)$. Then we have

$$0 < |x - c| < \delta \quad \Longrightarrow \quad 0 < \frac{f(x) - f(c)}{x - c} < 2f'(c)\,.$$

This implies that $x - c$ and $f(x) - f(c)$ will have the same sign when $0 < |x - c| < \delta$. It follows that

$$\begin{aligned} x \in (c - \delta, c) \quad &\Longrightarrow \quad x - c < 0 \quad \Longrightarrow \quad f(x) - f(c) < 0\,, \\ x \in (c, c + \delta) \quad &\Longrightarrow \quad x - c > 0 \quad \Longrightarrow \quad f(x) - f(c) > 0\,. \end{aligned}$$

Because $(c - \delta, c) \subset (a, b)$ and $(c, c + \delta) \subset (a, b)$, the result follows. $\square$

3. [10] Evaluate the following limit. (You may use theorems from class.)

$$\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4}\,.$$

Solution. Because the limit has a 0/0 indeterminant form, by the l'Hospital rule we obtain

$$\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4} = \lim_{x \to 2} \frac{3x^2}{2x} = \frac{3 \cdot 2^2}{2 \cdot 2} = \frac{12}{4} = 3\,.$$

$\square$

Second Solution. For every $x \neq \pm 2$ we have

$$\frac{x^3 - 8}{x^2 - 4} = \frac{x^2 + 2x + 4}{x + 2}\,.$$

Because the right-hand side above is continuous at $x = 2$, we have

$$\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4} = \lim_{x \to 2} \frac{x^2 + 2x + 4}{x + 2} = \frac{2^2 + 2 \cdot 2 + 4}{2 + 2} = \frac{12}{4} = 3\,.$$

□

4. [15] If $f(x) = \cosh(x) \equiv \frac{1}{2}(e^x + e^{-x})$ for every $x \in \mathbb{R}$ then for every $k \in \mathbb{N}$ we have
$$f^{(2k)}(x) = \cosh(x)\,, \qquad f^{(2k+1)}(x) = \sinh(x) \qquad \text{for every } x \in \mathbb{R}\,.$$
Use this fact to show that
$$\cosh(x) = \sum_{k=0}^{\infty} \frac{1}{(2k)!}\, x^{2k} \quad \text{for every } x \in \mathbb{R}\,.$$

Remark. There are many convergence tests that can be applied to show that the above series converges absolutely. For example, the Ratio and Roots Tests can do this. However, such convergence tests do not show that the series converges to $\cosh(x)$, which is what you are being asked to show!

Solution. Because for every $k \in \mathbb{N}$ we have
$$f^{(2k)}(0) = 1\,, \qquad f^{(2k+1)}(0) = 0\,,$$
the series is just the formal Taylor series for f centered at 0. The n^{th} partial sum of this series can be expressed as a Taylor polynomial approximation in two ways:
$$\sum_{k=0}^{n} \frac{1}{(2k)!}\, x^{2k} = T_0^{2n} \cosh(x) = T_0^{2n+1} \cosh(x)\,.$$
If we use the last expression then the Lagrange Remainder Theorem states that for every $x \in \mathbb{R}$ there exists some p between 0 and x such that
$$\cosh(x) = T_0^{2n+1} \cosh(x) + \frac{1}{(2n+2)!} \cosh(p) x^{2n+2}\,.$$
Hence, because $1 = \cosh(0) < \cosh(p) < \cosh(x)$, for every $x \in \mathbb{R}$ we have the bound
$$\left| \cosh(x) - \sum_{k=0}^{n} \frac{1}{(2k)!}\, x^{2k} \right| \leq \frac{1}{(2n+2)!} \cosh(x) |x|^{2n+2}\,.$$
Because factorials grow faster than exponentials, for every $x \in \mathbb{R}$ we have
$$\lim_{n\to\infty} \frac{1}{(2n+2)!} \cosh(x) |x|^{2n+2} = 0\,.$$
Therefore the sequence of partial sums converges to $\cosh(x)$. □

5. [10] Let $D \subset \mathbb{R}$. A function $f : D \to \mathbb{R}$ is said to be Hölder continuous of order $\alpha \in (0, 1]$ if there exists a $C > 0$ such that f satisfies the Hölder bound

$$|f(x) - f(y)| \leq C\,|x - y|^{\alpha} \qquad \text{for every } x, y \in D\,.$$

Prove that every such function is uniformly continuous over D.

Solution. By the ϵ-δ characterization of uniform continuity (the definition in the notes), we want to show that for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in D$ we have

$$|y - x| < \delta \qquad \Longrightarrow \qquad |f(y) - f(x)| < \epsilon\,.$$

Proof. Let $f : D \to \mathbb{R}$ satisfy the Hölder bound for some $\alpha \in (0, 1]$ and $C > 0$. Let $\epsilon > 0$. Pick $\delta = (\epsilon/C)^{\frac{1}{\alpha}}$. Then for every $x, y \in D$ we have

$$|x - y| < \delta \implies |f(x) - f(y)| \leq C\,|x - y|^{\alpha} < C\,\delta^{\alpha} = \epsilon\,.$$

Therefore f is uniformly continuous over D. □

Second Solution. By the sequence characterization of uniform continuity (the definition in the book), we want to show that for all sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}} \subset D$ we have

$$\lim_{n\to\infty} |y_n - x_n| = 0 \qquad \Longrightarrow \qquad \lim_{n\to\infty} \big|f(y_n) - f(x_n)\big| = 0\,.$$

Proof. Let $f : D \to \mathbb{R}$ satisfy the Hölder bound for some $\alpha \in (0, 1]$ and $C > 0$. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences in D such that

$$\lim_{n\to\infty} |y_n - x_n| = 0\,.$$

By the continuity of the power function we have

$$\lim_{n\to\infty} |y_n - x_n|^{\alpha} = \Big(\lim_{n\to\infty} |y_n - x_n|\Big)^{\alpha} = 0\,.$$

Then the Hölder bound implies that

$$\limsup_{n\to\infty} \big|f(y_n) - f(x_n)\big| \leq \limsup_{n\to\infty} C|y_n - x_n|^{\alpha} = C \lim_{n\to\infty} |y_n - x_n|^{\alpha} = 0\,.$$

But this implies that

$$\lim_{n\to\infty} \big|f(y_n) - f(x_n)\big| = 0\,.$$

Therefore f is uniformly continuous over D. □

6. [15] Prove that for every $x \in \mathbb{R}$ we have

$$1 + \tfrac{4}{3}x \leq (1+x)^{\frac{4}{3}} \,.$$

Solution. One approach to this problem uses the Monotonicity Theorem. Define $g(x) = (1+x)^{\frac{4}{3}} - 1 - \frac{4}{3}x$ for every $x \in \mathbb{R}$. Then g is continuously differentiable with

$$g'(x) = \tfrac{4}{3}\big[(1+x)^{\frac{1}{3}} - 1\big] \,.$$

Clearly, $g'(x) < 0$ for $x < 0$ while $g'(x) > 0$ for $x > 0$. By the Monotonicity Theorem, g is decreasing over $(-\infty, 0]$ and g is increasing over $[0, \infty)$. Therefore the global minimum of g over $\mathbb{R}$ is $g(0) = 0$. Hence, for every $x \in \mathbb{R}$ we have

$$(1+x)^{\frac{4}{3}} - 1 - \tfrac{4}{3}x = g(x) \geq g(0) = 0 \,.$$

The result follows. □

Second Solution. Another approach to this problem uses convexity ideas. Define $f(x) = (1+x)^{\frac{4}{3}}$ for every $x \in \mathbb{R}$. Then f is continuously differentiable over $\mathbb{R}$ with

$$f'(x) = \tfrac{4}{3}(1+x)^{\frac{1}{3}} \,,$$

and f is twice differentiable over $\mathbb{R} - \{-1\}$ with

$$f''(x) = \tfrac{4}{9}(1+x)^{-\frac{2}{3}} > 0 \,.$$

The Monotonicity Theorem applied to f' shows that f' is increasing over $\mathbb{R}$. The Convexity Characterization Theorem then implies that f is *strictly convex* over $\mathbb{R}$. This convexity implies that

$$f(x) - f(0) - f'(0)x \geq 0 \qquad \text{for every } x \in \mathbb{R} \,.$$

Therefore $(1+x)^{\frac{4}{3}} - 1 - \frac{4}{3}x \geq 0$ for every $x \in \mathbb{R}$. The result follows. □

Third Solution. Yet another approach uses the Lagrange Remainder Theorem. Define $f(x) = (1+x)^{\frac{4}{3}}$ for every $x \in \mathbb{R}$. Then f is twice differentiable over $x > -1$ with

$$f'(x) = \tfrac{4}{3}(1+x)^{\frac{1}{3}} \,, \qquad f''(x) = \tfrac{4}{9}(1+x)^{-\frac{2}{3}} \,.$$

By the Lagrange Remainder Theorem for every $x > -1$ there exists a p between 0 and x such that

$$f(x) - f(0) - f'(0)x = \tfrac{1}{2}f''(p)x^2 \,.$$

Hence, for every $x > -1$ we have

$$(1+x)^{\frac{4}{3}} - 1 - \tfrac{4}{3}x = \tfrac{4}{9}(1+p)^{-\frac{2}{3}}x^2 \geq 0 \,.$$

This gives the result for every $x > -1$. We can obtain the result for every $x \leq -1$ by observing that in that case $(1+x)^{\frac{4}{3}} \geq 0$ and $(1+x) \leq 0$, whereby

$$(1+x)^{\frac{4}{3}} - 1 - \tfrac{4}{3}x = (1+x)^{\frac{4}{3}} + \tfrac{1}{3} - \tfrac{4}{3}(1+x) \geq \tfrac{1}{3} > 0 \,.$$

Therefore the result follows for every $x \in \mathbb{R}$. □

7. [10] Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Let c be a limit point of D. Write negations of the following assertions.

(a) "For every sequence $\{x_k\}_{k\in\mathbb{N}} \subset D - \{c\}$ we have

$$\lim_{k\to\infty} |x_k - c| = 0 \quad \Longrightarrow \quad \lim_{k\to\infty} f(x_k) = \infty \,."$$

Solution. There exists a sequence $\{x_k\}_{k\in\mathbb{N}} \subset D - \{c\}$ such that

$$\lim_{k\to\infty} |x_k - c| = 0 \quad \text{and} \quad \liminf_{k\to\infty} f(x_k) < \infty \,.$$

□

Remark. The negation of "$\lim_{k\to\infty} f(x_k) = \infty$" is "$\liminf_{k\to\infty} f(x_k) < \infty$," not "$\lim_{k\to\infty} f(x_k) < \infty$."

(b) "For every $M \in \mathbb{R}$ there exists a $\delta > 0$ such that for every $x \in D$ we have

$$0 < |x - c| < \delta \quad \Longrightarrow \quad f(x) > M \,."$$

Solution. There exists $M \in \mathbb{R}$ such that for every $\delta > 0$ there exists $x \in D$ such that

$$0 < |x - c| < \delta \quad \text{and} \quad f(x) \le M \,.$$

□

8. [10] Show that the function $f(x) = x^2$ is not uniformly continuous over $\mathbb{R}$.

Solution. We can show the function $f(x) = x^2$ is not uniformly continuous over $\mathbb{R}$ by showing that it satisfies the negation of the sequence characterization of uniform continuity. This means we must find sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that

$$\lim_{n\to\infty} |y_n - x_n| = 0 \quad \text{and} \quad \limsup_{n\to\infty} \big|f(y_n) - f(x_n)\big| > 0\,.$$

Proof. Let $\{z_n\}_{n\in\mathbb{N}}$ be any sequence of positive numbers such that

$$\lim_{n\to\infty} z_n = 0\,.$$

For every $n \in \mathbb{N}$ define

$$x_n = \frac{1}{z_n}\,, \qquad\qquad y_n = \frac{1}{z_n} + z_n\,.$$

Then the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ satisfy

$$\lim_{n\to\infty} |y_n - x_n| = \lim_{n\to\infty} z_n = 0\,,$$

and

$$\begin{aligned}
\limsup_{n\to\infty} \big|f(y_n) - f(x_n)\big| &= \limsup_{n\to\infty} \left(\left(\frac{1}{z_n} + z_n\right)^2 - \left(\frac{1}{z_n}\right)^2 \right) \\
&= \limsup_{n\to\infty} \left(\frac{1}{z_n^2} + 2 + z_n^2 - \frac{1}{z_n^2} \right) \\
&= \limsup_{n\to\infty} \left(2 + z_n^2\right) = 2 > 0\,.
\end{aligned}$$

Therefore the function $f(x) = x^2$ is not uniformly continuous over $\mathbb{R}$. $\square$

Second Solution. We can show the function $f(x) = x^2$ is not uniformly continuous over $\mathbb{R}$ by showing that it satisfies the negation of the ϵ-δ characterization of uniform continuity. This means we must find an $\epsilon_o > 0$ such that for every $\delta > 0$ there exists $x_\delta, y_\delta \in D$ such that

$$|y_\delta - x_\delta| < \delta \quad \text{and} \quad \big|f(y_\delta) - f(x_\delta)\big| > \epsilon_o\,.$$

Any $\epsilon_o > 0$ can work. We will use $\epsilon_o = 1$.

Proof. Let $\delta > 0$. Pick x_δ and y_δ by

$$x_\delta = \frac{1}{\delta}\,, \qquad y_\delta = \frac{1}{\delta} + \frac{\delta}{2}\,.$$

Then $|y_\delta - x_\delta| = \frac{1}{2}\delta < \delta$ and

$$\big|f(y_\delta) - f(x_\delta)\big| = \left(\frac{1}{\delta} + \frac{\delta}{2}\right)^2 - \left(\frac{1}{\delta}\right)^2 = \frac{1}{\delta^2} + 1 + \frac{\delta^2}{4} - \frac{1}{\delta^2} = 1 + \frac{\delta^2}{4} > 1\,.$$

Therefore the function $f(x) = x^2$ is not uniformly continuous over $\mathbb{R}$. $\square$

9. [10] Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Suppose the equation $f'(x) = 0$ has at most one real solution. Prove that the equation $f(x) = 0$ has at most two real solutions.

Solution. Suppose that the equation $f'(x) = 0$ has at most one real solution while the equation $f(x) = 0$ has (at least) three real solutions $\{x_0, x_1, x_2\}$. Without loss of generality we can assume that

$$-\infty < x_0 < x_1 < x_2 < \infty \,.$$

Then for each $i = 1, 2$ we know that

- $f : [x_{i-1}, x_i] \to \mathbb{R}$ is differentiable (and hence is continuous),
- $f(x_{i-1}) = f(x_i) = 0$.

Rolle's Theorem then implies that for each $i = 1, 2$ there exists a point $p_i \in (x_{i-1}, x_i)$ such that $f'(p_i) = 0$. Because the intervals (x_0, x_1) and (x_1, x_2) are disjoint, the points p_1 and p_2 are distinct. The equation $f'(x) = 0$ therefore has at least two real solutions, which contradicts our starting supposition. □

Second Solution. There are two cases to consider: either $f'(x) = 0$ has no real solutions or it has exactly one real solution.

If $f'(x) = 0$ has no real solutions over $\mathbb{R}$ then by the Sign Dichotomy Theorem f' must be either negative or positive over $\mathbb{R}$. The Monotonicity Theorem then implies that f must be strictly monotonic (and hence one-to-one) over $\mathbb{R}$. The equation $f(x) = 0$ can thereby have at most one real solution.

If $f'(x) = 0$ has exactly one real solution c then by the Sign Dichotomy Theorem f' must be either negative or positive over each of the disjoint intervals

$$(-\infty, c) \,, \qquad (c, \infty) \,.$$

The Monotonicity Theorem then implies that f must be strictly monotonic (and hence one-to-one) over each of the two intervals

$$(-\infty, c] \,, \qquad [c, \infty) \,.$$

Therefore the equation $f(x) = 0$ can thereby have at most one solution in each of these intervals. Because the union of these intervals is $\mathbb{R}$, the equation $f(x) = 0$ can have at most two real solutions. □

Remark. The second solution rests upon the Sign Dichotomy Theorem and the Monotonicity Theorem. This is heavier machinery than was used in the first solution, which rests only upon Rolle's Theorem. Indeed, the Monotonicity Theorem rests upon the Mean-Value Theorem, the proof of which rests upon Rolle's Theorem.

Exercise. Modify the above proofs to prove the following fact. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Let $n \in \mathbb{Z}_+$. Suppose that the equation $f'(x) = 0$ has at most n real solutions. Show that the equation $f(x) = 0$ has at most $n + 1$ real solutions.