Second In-Class Exam Solutions: Math 410
Section 0401, Professor Levermore
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1. [10] Give a counterexample to each of the following assertions.

(a) If $f : \mathbb{R} \to \mathbb{R}$ is monotonic and one-to-one then it is also continuous.

Solution. There are many counterexamples. The simplest ones have a single jump discontinuity somewhere. For example, consider the function f defined by

$$f(x) = \begin{cases} x & \text{for } x < 0\,, \\ x + 1 & \text{for } x \geq 0\,. \end{cases}$$

This function is clearly increasing and one-to-one, but has a jump discontinuity at $x = 0$ because

$$\lim_{x \to 0^-} f(x) = 0 \neq 1 = \lim_{x \to 0^+} f(x)\,.$$

$\square$

(b) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable then its derivative $f' : \mathbb{R} \to \mathbb{R}$ is continuous.

Solution. There are many counterexamples. The one we discussed in class was

$$f(x) = \begin{cases} x^2 \cos\left(\dfrac{1}{x}\right) & \text{for } x \neq 0\,, \\ 0 & \text{for } x = 0\,. \end{cases}$$

This function is clearly differentiable at every $x \neq 0$ with

$$f'(x) = 2x \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right) \quad \text{for } x \neq 0\,.$$

Moreover, it is differentiable at $x = 0$ with

$$\begin{aligned} f'(0) &= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} \\ &= \lim_{x \to 0} x \cos\left(\frac{1}{x}\right) = 0\,. \end{aligned}$$

Hence, f is differentiable over $\mathbb{R}$. However, because

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) \quad \text{does not exist}\,,$$

while

$$\lim_{x \to 0} 2x \cos\left(\frac{1}{x}\right) = 0\,,$$

it follows that

$$\lim_{x \to 0} f'(x) \quad \text{does not exist}\,.$$

Therefore f' is not continuous at $x = 0$. $\square$

2. [10] Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Prove it is continuous.

Solution. This statement is *true*. Let $c \in \mathbb{R}$ be arbitrary. Because f is differentiable at c we know that

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) .$$

Because for every $x \in \mathbb{R}$ such that $x \neq c$ one has

$$f(x) = f(c) + \frac{f(x) - f(c)}{x - c} \, (x - c) ,$$

it follows from the algebraic properties of limits that

$$\begin{aligned} \lim_{x \to c} f(x) &= \lim_{x \to c} f(c) + \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} (x - c) \\ &= f(c) + f'(c) \cdot 0 = f(c) . \end{aligned}$$

Hence, f is continuous at c. Because $c \in \mathbb{R}$ was arbitrary, f is continuous over $\mathbb{R}$. □

Remark. The facts

$$\lim_{x \to c} f(c) = f(c) , \quad \text{and} \quad \lim_{x \to c} (x - c) = 0 ,$$

were used above without fanfare. You do not have to give proofs of such elementary facts unless you are explicitly asked to do so.

3. [15] Show that

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \, x^{2k} \quad \text{for every } x \in \mathbb{R} .$$

Remark. It is not enough to simply show that the series converges by some convergence test. For example, the ratio test shows the series converges for every $x \in \mathbb{R}$, but does not show that it converges to $\cos(x)$.

Solution. Let $f(x) = \cos(x)$ for every $x \in \mathbb{R}$. Then for every $k \in \mathbb{N}$ one has

$$f^{(2k)}(x) = (-1)^k \cos(x) , \qquad f^{(2k+1)}(x) = (-1)^{k+1} \sin(x) .$$

Because

$$f^{(2k)}(0) = (-1)^k , \qquad f^{(2k+1)}(0) = 0 ,$$

the series is just the formal Taylor series for f centered at 0. Moreover, we see that the n^{th} partial sum can be expressed as a Taylor polynomial approximation in two ways:

$$\sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} \, x^{2k} = T_0^{(2n)} \cos(x) = T_0^{(2n+1)} \cos(x) .$$

Using the last expression, the Lagrange Remainder Theorem states that for every $x \in \mathbb{R}$

$$\cos(x) = T_0^{(2n+1)} \cos(x) + \frac{(-1)^{n+1}}{(2n+2)!} \cos(p) x^{2n+2} ,$$

for some p between 0 and x. Hence, for every $x \in \mathbb{R}$

$$\left| \cos(x) - \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} \, x^{2k} \right| \leq \frac{1}{(2n+2)!} \, |x|^{2n+2} .$$

Because for every $x \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{(2n+2)!}\,|x|^{2n+2} = 0\,,$$

the sequence of partial sums therefore converges to $\cos(x)$. $\square$

4. [10] Prove that for every $x > -1$ one has

$$(1+x)^{\frac{3}{4}} \leq 1 + \tfrac{3}{4}x\,.$$

Solution. One approach to this problem is to use the Lagrange Remainder Theorem. Define $f(x) = (1+x)^{\frac{3}{4}}$ for every $x > -1$. Then

$$f'(x) = \tfrac{3}{4}(1+x)^{-\frac{1}{4}}\,, \qquad f''(x) = -\tfrac{3}{16}(1+x)^{-\frac{5}{4}}\,.$$

For every $x > -1$ the Lagrange Remainder Theorem implies there exists a p between 0 and x such that

$$f(x) = f(0) + f'(0)x + \tfrac{1}{2}f''(p)x^2\,.$$

Hence, becasue $(1+p)^{-\frac{5}{4}} > 0$ we obtain

$$(1+x)^{\frac{3}{4}} = 1 + \tfrac{3}{4}x - \tfrac{3}{32}(1+p)^{-\frac{5}{4}}x^2 \leq 1 + \tfrac{3}{4}x\,,$$

which is the desired result. $\square$

Alternative Solution. Another approach to this problem is to use the Monotonicity Theorem. Define $g(x) = (1+x)^{\frac{3}{4}} - 1 - \frac{3}{4}x$ for every $x > -1$. Then

$$g'(x) = \tfrac{3}{4}\big[(1+x)^{-\frac{1}{4}} - 1\big]\,.$$

Clearly, $g'(x) > 0$ for $x \in (-1,0)$ while $g'(x) < 0$ for $x \in (0,\infty)$. By the Monotonicity Theorem, g is increasing over $x \in (-1,0]$ and g is decreasing over $[0,\infty)$. Therefore $x = 0$ is a global maximizer of g over $(-1,\infty)$, and $g(0) = 0$ is the maximum of g over $(-1,\infty)$. Hence, for every $x > -1$ we have

$$(1+x)^{\frac{3}{4}} - 1 - \tfrac{3}{4}x = g(x) \leq g(0) = 0\,,$$

which is the desired result. $\square$

Another Alternative Solution. You can also approach this problem by using the fact that the function $h(t) = t^{\frac{1}{4}}$ is increasing over $t > 0$. The result will then follow once you show that

$$(1+x)^3 \leq (1+\tfrac{3}{4}x)^4\,.$$

The binomial expansion yields

$$\begin{aligned}(1+\tfrac{3}{4}x)^4 - (1+x)^3 &= 1 + 4\tfrac{3}{4}x + 6\tfrac{3^2}{4^2}x^2 + 4\tfrac{3^3}{4^3}x^3 + \tfrac{3^4}{4^4}x^4 \\ &\quad - 1 - 3x - 3x^2 - x^3 \\ &= \tfrac{3}{8}x^2 + \tfrac{11}{16}x^3 + \tfrac{81}{256}x^4 = \tfrac{1}{256}x^2\big(96 + 176x + 81x^2\big)\,.\end{aligned}$$

There are several ways to show that $96 + 176x + 81x^2 > 0$. For example, you can show that $96+176x+81x^2$ has no real roots because its descriminant is $96\cdot 81 - 88^2 = 32 > 0$. You can also use techniques from calculus.

5. [10] Evaluate the following limit. Give your reasoning. (You may use theorems we have proved in class.)

$$\lim_{x\to 2} \frac{x^3-8}{x^2-4}\,.$$

Solution. For every $x \neq 2$ one has

$$\frac{x^3-8}{x^2-4} = \frac{x^2+2x+4}{x+2}\,.$$

Because the right-hand side above is continuous over $(-2,\infty)$, one has

$$\lim_{x\to 2} \frac{x^3-8}{x^2-4} = \lim_{x\to 2} \frac{x^2+2x+4}{x+2} = \frac{4+4+4}{2+2} = 3\,.$$

$\square$

Alternative Solution. Because the limit has a $0/0$ indeterminant form, by the l'Hopital rule we obtain

$$\lim_{x\to 2} \frac{x^3-8}{x^2-4} = \lim_{x\to 2} \frac{3x^2}{2x} = \frac{3\cdot 4}{2\cdot 2} = 3\,.$$

6. [15] Use the ϵ-δ criterion to prove that

$$\lim_{x\to 2} \frac{x^3-1}{x-1} = 7\,.$$

Solution. You must show that for every $\epsilon > 0$ and $x \neq 1$ there exists $\delta > 0$ such that

$$0 < |x-2| < \delta \qquad \Longrightarrow \qquad \left| \frac{x^3-1}{x-1} - 7 \right| < \epsilon\,.$$

You should use the fact that for every $x \neq 1$ one has

$$\frac{x^3-1}{x-1} = x^2+x+1\,.$$

Let $\epsilon > 0$. You can, for example, pick $\delta = \epsilon/(5+\frac{1}{5}\epsilon)$. Then for every $x \neq 1$

$$\begin{aligned}
0 < |x-2| < \delta \qquad \Longrightarrow \qquad \left| \frac{x^3-1}{x-1} - 7 \right| &= \left| x^2+x-6 \right| = |x-2||x+3| \\
&\leq |x-2|\big(5+|x-2|\big) \\
&< \delta(5+\delta) = \frac{\epsilon}{5+\frac{1}{5}\epsilon}\left(5 + \frac{\epsilon}{5+\frac{1}{5}\epsilon}\right) < \epsilon\,,
\end{aligned}$$

which is what had to be shown. $\square$

Remark. There are many other choices of δ that could be made. For example, you could pick

$$\delta = \min\left\{1\,,\, \tfrac{1}{6}\epsilon\right\}, \qquad \text{or} \qquad \delta = \frac{\epsilon}{\frac{5}{2} + \sqrt{\frac{25}{4}+\epsilon}}\,.$$

7. [10] Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Let $n \in \mathbb{N}$. Suppose the following equation has at most n solutions:
$$f'(x) = 0\,, \qquad x \in \mathbb{R}\,.$$
Show the following equation has at most $n+1$ solutions:
$$f(x) = 0\,, \qquad x \in \mathbb{R}\,.$$

Solution. Suppose that the equation $f'(x) = 0$ has at most n solutions while the equation $f(x) = 0$ has $n+2$ solutions $\{x_i\}_{i=0}^{n+1}$. Without loss of generality we can assume these points are labeled so that
$$-\infty < x_0 < x_1 < \cdots < x_n < x_{n+1} < \infty\,.$$
Then for each $i = 1, \cdots, n+1$ one knows that
- $f : [x_{i-1}, x_i] \to \mathbb{R}$ is differentiable (and hence continuous),
- $f(x_{i-1}) = f(x_i) = 0$.

Rolle's Theorem then implies that for each $i = 1, \cdots, n+1$ there exists a point $p_i \in (x_{i-1}, x_i)$ such that $f'(p_i) = 0$. Because the $n+1$ intervals (x_{i-1}, x_i) are disjoint, the points p_i are distinct. The equation $f'(x) = 0$ therefore has at least $n+1$ solutions, which contradicts our starting supposition. □

Alternative Solution. Suppose $f'(x) = 0$ has exactly m solutions $\{c_i\}_{i=1}^{m}$, where $m \leq n$. Without loss of generality we can assume these m critical points are labeled so that
$$-\infty < c_1 < c_2 < \cdots < c_{m-1} < c_m < \infty\,.$$
By the Dichotomy Theorem f' must be either negative or positive over each of the $m+1$ disjoint intervals
$$(-\infty, c_1)\,, \quad (c_1, c_2)\,, \quad \cdots \quad (c_{m-1}, c_m)\,, \quad (c_m, \infty)\,.$$
By the Monotonicity Theorem f must be monotonic (and hence one-to-one) over each of the $m+1$ intervals
$$(-\infty, c_1]\,, \quad [c_1, c_2]\,, \quad \cdots \quad [c_{m-1}, c_m]\,, \quad [c_m, \infty)\,.$$
The equation $f(x) = 0$ can therefore have at most one solution in each of these $m+1$ intervals. Because the union of these intervals is $\mathbb{R}$, the equation $f(x) = 0$ can have at most $m+1$ solutions. The result follows because $m+1 \leq n+1$. □

Remark. The alternative solution rests on the Dichotomy Theorem and the Monotonicity Theorem. This machinery is much heavier than that used in the first solution, which rests only on Rolle's Theorem. Indeed, the proof of the Monotonicity Theorem rests on the Mean-Value Theorem, the proof of which rests on Rolle's Theorem.

8. [10] Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be uniformly continuous over D. Let $\{x_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence contained in D. Show that $\{f(x_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence.

Solution. Let $\epsilon > 0$. Because $f : (a, b) \to \mathbb{R}$ is uniformly continuous over (a, b), there exists a $\delta > 0$ such that for every $x, y \in D$ one has

$$|x - y| < \delta \quad \Longrightarrow \quad \big|f(x) - f(y)\big| < \epsilon\,.$$

Because $\{x_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence, there exists an $N \in \mathbb{N}$ such that for every $k, l \in \mathbb{N}$ one has

$$k, l > N \quad \Longrightarrow \quad |x_k - x_l| < \delta\,.$$

Hence, for every $k, l \in \mathbb{N}$ one has

$$\begin{aligned} k, l > N \quad &\Longrightarrow \quad |x_k - x_l| < \delta \\ &\Longrightarrow \quad \big|f(x_k) - f(x_l)\big| < \epsilon\,. \end{aligned}$$

Therefore $\{f(x_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence. □

9. [10] Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Let c be a limit point of D. Write negations of the following assertions.

(a) "For every sequence $\{x_k\}_{k\in\mathbb{N}} \subset D - \{c\}$ one has

$$\lim_{k\to\infty} |x_k - c| = 0 \quad \Longrightarrow \quad \lim_{k\to\infty} f(x_k) = -\infty\,."$$

Solution. There exists a sequence $\{x_k\}_{k\in\mathbb{N}} \subset D - \{c\}$ such that

$$\lim_{k\to\infty} |x_k - c| = 0 \quad \text{and} \quad \limsup_{k\to\infty} f(x_k) > -\infty\,.$$

(b) "For every $M \in \mathbb{R}$ there exists a $\delta > 0$ such that for every $x \in D$ one has

$$0 < |x - c| < \delta \quad \Longrightarrow \quad f(x) < M\,."$$

Solution: There exists $M \in \mathbb{R}$ such that for every $\delta > 0$ there exists $x \in D$ such that

$$0 < |x - c| < \delta \quad \text{and} \quad f(x) \geq M\,.$$