**Second In-Class Exam Solutions**
**Math 410, Professor David Levermore**
**Thursday, 2 November 2017**

1. [10] Give (with reasoning) a counterexample to each of the following false assertions.
   (a) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and increasing over $\mathbb{R}$ then $f' > 0$ over $\mathbb{R}$.
   (b) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable then its derivative $f' : \mathbb{R} \to \mathbb{R}$ is continuous.

**Solution (a).** There are many counterexamples. One that was presented in class is

$$f(x) = x^3 .$$

This function is clearly continuously differentiable over $\mathbb{R}$ with $f'(x) = 3x^2$. We see that $f'(0) = 0$ and that $f'(x) > 0$ for every $x \neq 0$.

Because $f'(x) > 0$ over $(-\infty, 0)$ and over $(0, \infty)$, the Monotonicity Theorem says that $f$ is increasing over $(-\infty, 0]$ and over $[0, \infty)$. Because $f$ is increasing over $(-\infty, 0]$ and over $[0, \infty)$, it is increasing over $\mathbb{R}$.

Therefore $f : \mathbb{R} \to \mathbb{R}$ is differentiable and increasing over $\mathbb{R}$ but it is false that $f' > 0$ because $f'(0) = 0$. $\square$

**Solution (b).** There are many counterexamples. One that was presented in class is

$$f(x) = \begin{cases} x^2 \cos\left(\dfrac{1}{x}\right) & \text{for } x \neq 0 , \\ 0 & \text{for } x = 0 . \end{cases}$$

This function is clearly continuously differentiable at every $x \neq 0$ with

$$f'(x) = 2x \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right) \quad \text{for } x \neq 0 .$$

Moreover, it is differentiable at $x = 0$ with

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} x \cos\left(\frac{1}{x}\right) = 0 .$$

Hence, $f$ is differentiable over $\mathbb{R}$. However, because

$$\lim_{x \to 0} 2x \cos\left(\frac{1}{x}\right) = 0 ,$$

while

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) \quad \text{does not exist} ,$$

it follows from the formula for $f'(x)$ given earlier that

$$\lim_{x \to 0} f'(x) = \lim_{x \to 0} \sin\left(\frac{1}{x}\right) \quad \text{does not exist} .$$

Therefore $f'$ is not continuous at $x = 0$. $\square$

2. [10] Let $f : (a, b) \to \mathbb{R}$ be differentiable at a point $c \in (a, b)$ with $f'(c) < 0$. Show that there exists a $\delta > 0$ such that

$$x \in (c-\delta, c) \subset (a, b) \implies f(x) > f(c)\,,$$
$$x \in (c, c+\delta) \subset (a, b) \implies f(c) > f(x)\,,$$

**Remark.** The problem is asking you to prove the Transversality Lemma.

**Solution.** Because $f$ is differentiable at $c$ we know that

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}\,.$$

By the $\epsilon$-$\delta$ definition of limit, this means that for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x \in (a, b)$ we have

$$0 < |x - c| < \delta \qquad \implies \qquad \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \epsilon\,.$$

Because $f'(c) < 0$ we may take $\epsilon = -f'(c)$ above to conclude that there exists $\delta > 0$ such that for every $x \in (a, b)$ we have

$$0 < |x - c| < \delta \qquad \implies \qquad \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < -f'(c)\,.$$

Because $c \in (a, b)$ we may assume that $\delta$ is small enough so that $(c-\delta, c+\delta) \subset (a, b)$. Then we have

$$0 < |x - c| < \delta \qquad \implies \qquad 2f'(c) < \frac{f(x) - f(c)}{x - c} < 0\,.$$

This implies that $x - c$ and $f(x) - f(c)$ will have opposite signs when $0 < |x - c| < \delta$. It follows that

$$x \in (c-\delta, c) \quad \implies \quad x - c < 0 \quad \implies \quad f(x) - f(c) > 0\,,$$
$$x \in (c, c+\delta) \quad \implies \quad x - c > 0 \quad \implies \quad f(x) - f(c) < 0\,.$$

Because $(c - \delta, c) \subset (a, b)$ and $(c, c + \delta) \subset (a, b)$, the result follows. □

3. [10] Evaluate the following limit. (You may use theorems from class.)

$$\lim_{x\to 3} \frac{x^4 - 81}{x^3 - 27}\,.$$

**Solution.** The limit has a 0/0 indeterminant form. By the l'Hospital rule we obtain

$$\lim_{x\to 3} \frac{x^4 - 81}{x^3 - 27} = \lim_{x\to 3} \frac{4x^3}{3x^2} = \frac{4 \cdot 3^3}{3 \cdot 3^2} = \frac{4 \cdot 27}{27} = 4\,.$$

□

**Second Solution.** For every $x \neq 3$ we have

$$\frac{x^4 - 81}{x^3 - 27} = \frac{(x-3)(x^3 + 3x^2 + 9x + 27)}{(x-3)(x^2 + 3x + 9)} = \frac{x^3 + 3x^2 + 9x + 27}{x^2 + 3x + 9}\,.$$

Because the right-hand side above is continuous at $x = 3$, we have

$$\lim_{x\to 3} \frac{x^4 - 81}{x^3 - 27} = \lim_{x\to 3} \frac{x^3 + 3x^2 + 9x + 27}{x^2 + 3x + 9} = \frac{3^3 + 3 \cdot 3^2 + 9 \cdot 3 + 27}{3^2 + 3 \cdot 3 + 9} = \frac{4 \cdot 27}{27} = 4\,.$$

□

4. [15] If $f(x) = \sin(x)$ for every $x \in \mathbb{R}$ then for every $k \in \mathbb{N}$ we have

$$f^{(2k)}(x) = (-1)^k \sin(x)\,, \qquad f^{(2k+1)}(x) = (-1)^k \cos(x) \qquad \text{for every } x \in \mathbb{R}\,.$$

Use this fact to show that

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1} \quad \text{for every } x \in \mathbb{R}\,.$$

**Remark.** There are many convergence tests that can be applied to show that the above series converges absolutely. For example, if we apply the Ratio Test then because for every $x \in \mathbb{R}$ we have

$$\lim_{k\to\infty} \frac{\dfrac{1}{(2k+3)!}\,|x|^{2k+3}}{\dfrac{1}{(2k+1)!}\,|x|^{2k+1}} = \lim_{k\to\infty} \frac{|x|^2}{(2k+2)(2k+3)} = 0\,,$$

we conclude for every $x \in \mathbb{R}$ that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1} \qquad \text{converges absolutely}\,.$$

However, such convergence tests do not show that the series converges to $\sin(x)$, which is what you are being asked to show!

**Solution.** Because for every $k \in \mathbb{N}$ we have

$$f^{(2k)}(0) = 0\,, \qquad f^{(2k+1)}(0) = (-1)^k\,,$$

the series is just the formal Taylor series for $f$ centered at 0. The $n^{\text{th}}$ partial sum of this series can be expressed as a Taylor polynomial approximation in two ways:

$$\sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1} = T_0^{2n+1} \sin(x) = T_0^{2n+2} \sin(x)\,.$$

If we use the last expression then the Lagrange Remainder Theorem states that for every $x \in \mathbb{R}$ there exists some $p$ between 0 and $x$ such that

$$\sin(x) = T_0^{2n+2} \sin(x) + \frac{(-1)^{n+1}}{(2n+3)!} \cos(p) x^{2n+3}\,.$$

Hence, because $|\cos(p)| \leq 1$ for every $x \in \mathbb{R}$, we have the bound

$$\left| \sin(x) - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1} \right| \leq \frac{1}{(2n+3)!}\, |x|^{2n+3}\,.$$

Because factorials grow faster than exponentials, for every $x \in \mathbb{R}$ we have

$$\lim_{n\to\infty} \frac{1}{(2n+3)!}\, |x|^{2n+3} = 0\,.$$

Therefore the sequence of partial sums converges to $\sin(x)$. □

5. [10] Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be uniformly continuous over $D$. Let $\{x_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence contained in $D$. Show that $\{f(x_k)\}_{k\in\mathbb{N}}$ is a convergent sequence.

**Solution.** Because every Cauchy sequence is convergent, it is enough to show that $\{f(x_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence. This means we have to show that for every $\epsilon > 0$ there exists an $n_\epsilon \in \mathbb{N}$ such that

$$m, n \geq n_\epsilon \quad \Longrightarrow \quad |f(x_m) - f(x_n)| < \epsilon \,.$$

**Proof.** Let $\epsilon > 0$. Because $f : D \to \mathbb{R}$ is uniformly continuous over $D$, there exists $\delta_\epsilon > 0$ such that for every $x, y \in D$ we have

$$|x - y| < \delta_\epsilon \quad \Longrightarrow \quad |f(x) - f(y)| < \epsilon \,.$$

Because $\{x_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence, there exists an $n_\epsilon \in \mathbb{N}$ such that

$$m, n \geq n_\epsilon \quad \Longrightarrow \quad |x_m - x_n| < \delta_\epsilon \,.$$

Because $\{x_k\}_{k\in\mathbb{N}}$ is contained in $D$, it follows that

$$|x_m - x_n| < \delta_\epsilon \quad \Longrightarrow \quad |f(x_m) - f(x_n)| < \epsilon \,.$$

By combining the last two implications we see that

$$m, n \geq n_\epsilon \quad \Longrightarrow \quad |f(x_m) - f(x_n)| < \epsilon \,.$$

Therefore the sequence $\{f(x_k)\}_{k\in\mathbb{N}}$ is Cauchy and thereby is convergent. □

6. [10] Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Let $c$ be a limit point of $D$. Write negations of the following assertions.

(a) "For every sequence $\{x_k\}_{k\in\mathbb{N}} \subset D - \{c\}$ we have

$$\lim_{k\to\infty} |x_k - c| = 0 \quad \Longrightarrow \quad \lim_{k\to\infty} f(x_k) = -\infty \,."$$

(b) "For every $M \in \mathbb{R}$ there exists a $\delta > 0$ such that for every $x \in D$ we have

$$0 < |x - c| < \delta \quad \Longrightarrow \quad f(x) > M \,."$$

**Solution (a).** There exists a sequence $\{x_k\}_{k\in\mathbb{N}} \subset D - \{c\}$ such that

$$\lim_{k\to\infty} |x_k - c| = 0 \quad \text{and} \quad \limsup_{k\to\infty} f(x_k) > -\infty \,.$$

□

**Remark.** The negation of "$\lim_{k\to\infty} f(x_k) = -\infty$" is "$\limsup_{k\to\infty} f(x_k) > -\infty$," not "$\lim_{k\to\infty} f(x_k) > -\infty$."

**Solution (b).** There exists $M \in \mathbb{R}$ such that for every $\delta > 0$ there exists $x \in D$ such that

$$0 < |x - c| < \delta \quad \text{and} \quad f(x) \leq M \,.$$

□

7. [15] Prove that for every nonzero $x \in \mathbb{R}$ we have

$$1 + \tfrac{8}{7}x < (1+x)^{\frac{8}{7}}\,.$$

**Solution.** One approach to this problem uses the Monotonicity Theorem. Define $g(x) = (1+x)^{\frac{8}{7}} - 1 - \frac{8}{7}x$ for every $x \in \mathbb{R}$. Then $g$ is continuously differentiable with

$$g'(x) = \tfrac{8}{7}\big[(1+x)^{\frac{1}{7}} - 1\big]\,.$$

Clearly, $g'(x) < 0$ for $x < 0$ while $g'(x) > 0$ for $x > 0$. By the Monotonicity Theorem, $g$ is decreasing over $(-\infty, 0]$ and $g$ is increasing over $[0, \infty)$. Therefore the global minimum of $g$ over $\mathbb{R}$ is $g(0) = 0$. Hence, for every $x \in \mathbb{R}$ we have

$$(1+x)^{\frac{8}{7}} - 1 - \tfrac{8}{7}x = g(x) \geq g(0) = 0\,.$$

The result follows. □

**Second Solution.** Another approach to this problem uses convexity ideas. Define $f(x) = (1+x)^{\frac{8}{7}}$ for every $x \in \mathbb{R}$. Then $f$ is continuously differentiable over $\mathbb{R}$ with

$$f'(x) = \tfrac{8}{7}(1+x)^{\frac{1}{7}}\,,$$

and $f$ is twice differentiable over $\mathbb{R} - \{-1\}$ with

$$f''(x) = \tfrac{8}{49}(1+x)^{-\frac{6}{7}} > 0\,.$$

The Monotonicity Theorem applied to $f'$ shows that $f'$ is increasing over $(-\infty, -1]$ and over $[-1, \infty)$, whereby it is increasing over $\mathbb{R}$. The Convexity Characterization Theorem then implies that $f$ is *strictly convex* over $\mathbb{R}$. This convexity implies that

$$f(x) - f(0) - f'(0)x \geq 0 \qquad \text{for every } x \in \mathbb{R}\,.$$

Therefore $(1+x)^{\frac{8}{7}} - 1 - \frac{8}{7}x \geq 0$ for every $x \in \mathbb{R}$. The result follows. □

**Third Solution.** Yet another approach uses the Lagrange Remainder Theorem. Define $f(x) = (1+x)^{\frac{8}{7}}$ for every $x \in \mathbb{R}$. Then $f$ is twice differentiable over $x > -1$ with

$$f'(x) = \tfrac{8}{7}(1+x)^{\frac{1}{7}}\,, \qquad f''(x) = \tfrac{8}{49}(1+x)^{-\frac{6}{7}}\,.$$

By the Lagrange Remainder Theorem for every $x > -1$ there exists a $p$ between 0 and $x$ such that

$$f(x) = f(0) + f'(0)x + \tfrac{1}{2}f''(p)x^2\,.$$

Hence, for every $x > -1$ we have

$$(1+x)^{\frac{8}{7}} - 1 - \tfrac{8}{7}x = \tfrac{4}{49}(1+p)^{-\frac{2}{3}}x^2 \geq 0\,.$$

This gives the result for every $x > -1$. We can obtain the result for every $x \leq -1$ by observing that in that case $(1+x)^{\frac{8}{7}} \geq 0$ and $(1+x) \leq 0$, whereby

$$(1+x)^{\frac{8}{7}} - 1 - \tfrac{8}{7}x = (1+x)^{\frac{8}{7}} + \tfrac{1}{7} - \tfrac{8}{7}(1+x) \geq \tfrac{1}{7} > 0\,.$$

Therefore the result follows for every $x \in \mathbb{R}$. □

8. [10] Show that the function $f(x) = 1/x$ is not uniformly continuous over $\mathbb{R}_+$.

**Solution.** We can show the function $f(x) = 1/x$ is not uniformly continuous over $\mathbb{R}_+$ by showing that it satisfies the negation of the sequence characterization of uniform continuity. This means we must find sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that

$$\lim_{n\to\infty} |y_n - x_n| = 0 \quad \text{and} \quad \limsup_{n\to\infty} \big|f(y_n) - f(x_n)\big| > 0\,.$$

**Proof.** Let $\{z_n\}_{n\in\mathbb{N}}$ be any sequence of positive numbers such that

$$\lim_{n\to\infty} z_n = 0\,.$$

For every $n \in \mathbb{N}$ define (here we make different choices than in the notes)

$$x_n = z_n^{\frac{1}{2}}\,, \qquad\qquad y_n = z_n^{\frac{1}{2}} + z_n\,.$$

Then the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are contained in $\mathbb{R}_+$ and satisfy

$$\lim_{n\to\infty} |y_n - x_n| = \lim_{n\to\infty} z_n = 0\,,$$

and

$$\begin{aligned}\lim_{n\to\infty} \big|f(y_n) - f(x_n)\big| &= \lim_{n\to\infty} \left(\frac{1}{x_n} - \frac{1}{y_n}\right) = \lim_{n\to\infty} \frac{y_n - x_n}{x_n\, y_n} \\ &= \lim_{n\to\infty} \frac{z_n}{z_n + z_n^{\frac{3}{2}}} = \lim_{n\to\infty} \frac{1}{1 + z_n^{\frac{1}{2}}} = 1 \neq 0\,.\end{aligned}$$

Therefore the function $f(x) = 1/x$ is not uniformly continuous over $\mathbb{R}_+$. □

**Second Solution.** We can show the function $f(x) = 1/x$ is not uniformly continuous over $\mathbb{R}_+$ by showing that it satisfies the negation of the $\epsilon$-$\delta$ characterization of uniform continuity. This means we must find an $\epsilon_o > 0$ such that for every $\delta > 0$ there exists $x_\delta, y_\delta \in \mathbb{R}_+$ such that

$$|y_\delta - x_\delta| < \delta \quad \text{and} \quad \big|f(y_\delta) - f(x_\delta)\big| \geq \epsilon_o\,.$$

Any $\epsilon_o > 0$ can work. We will use $\epsilon_o = \frac{1}{2}$.

**Proof.** Let $\delta > 0$. Pick $x_\delta$ and $y_\delta$ by

$$x_\delta = \frac{\delta}{1+\delta}\,, \qquad y_\delta = \frac{2\delta}{1+\delta}\,.$$

Then $0 < x_\delta < y_\delta$,

$$|y_\delta - x_\delta| = \frac{2\delta}{1+\delta} - \frac{\delta}{1+\delta} = \frac{\delta}{1+\delta} < \delta\,,$$

and

$$\big|f(y_\delta) - f(x_\delta)\big| = \frac{1}{x_\delta} - \frac{1}{y_\delta} = \frac{1+\delta}{\delta} - \frac{1+\delta}{2\delta} = \frac{1+\delta}{2\delta} > \tfrac{1}{2}\,.$$

Therefore the function $f(x) = 1/x$ is not uniformly continuous over $\mathbb{R}_+$. □

9. [10] Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Suppose the equation $f'(x) = 0$ has at most five real solutions. Prove that the equation $f(x) = 0$ has at most six real solutions.

**Solution.** Suppose that the equation $f(x) = 0$ has (at least) seven real solutions $\{x_0, x_1, x_2, x_3, x_4, x_5, x_6\}$. Without loss of generality we can assume that

$$-\infty < x_0 < x_1 < x_2 < x_3 < x_4 < x_5 < x_6 < \infty .$$

Then for each $i = 1, 2, \cdots, 6$ we know that

- $f : [x_{i-1}, x_i] \to \mathbb{R}$ is differentiable (and hence is continuous),
- $f(x_{i-1}) = f(x_i) = 0$.

Rolle's Theorem then implies that for each $i = 1, 2, \cdots, 6$ there exists a point $p_i \in (x_{i-1}, x_i)$ such that $f'(p_i) = 0$. Because the intervals $\{(x_{i-1}, x_i)\}_{i=1}^{6}$ are disjoint, the points $\{p_i\}_{i=1}^{6}$ are distinct. Therefore equation $f'(x) = 0$ has at least six real solutions, which shows that it has more than five real solutions. □

**Second Solution.** Suppose that $f'(x) = 0$ has exactly $n$ real solutions for some $n \in \{0, 1, 2, 3, 4, 5\}$.

If $n = 0$ then $f'(x) = 0$ has no real solutions over $\mathbb{R}$. By the Sign Dichotomy Theorem $f'$ must be either negative or positive over $\mathbb{R}$. The Monotonicity Theorem then implies that $f$ must be strictly monotonic (and hence one-to-one) over $\mathbb{R}$. The equation $f(x) = 0$ can thereby have at most one real solution.

If $n > 0$ then $f'(x) = 0$ has exactly $n$ real solutions $\{c_1, \cdots, c_n\}$ for some $n \in \{1, 2, 3, 4, 5\}$. Without loss of generality we can assume that

$$-\infty < c_1 < c_2 < \cdots < c_n < \infty .$$

Then by the Sign Dichotomy Theorem $f'$ must be either negative or positive over each of the $n + 1$ disjoint intervals

$$(-\infty, c_1) , \quad (c_1, c_2) , \cdots , \quad (c_n, \infty) .$$

The Monotonicity Theorem then implies that $f$ must be strictly monotonic (and hence one-to-one) over each of the $n + 1$ intervals

$$(-\infty, c_1] , \quad [c_1, c_2] , \cdots , \quad [c_n, \infty) .$$

Therefore the equation $f(x) = 0$ can thereby have at most one solution in each of these intervals. Because the union of these intervals is $\mathbb{R}$, the equation $f(x) = 0$ can have at most $n + 1$ real solutions. Because $n$ is at most 5, the equation $f(x) = 0$ can have at most 6 real solutions. □

**Remark.** The second solution rests upon the Sign Dichotomy Theorem and the Monotonicity Theorem. This is heavier machinery than was used in the first solution, which rests only upon Rolle's Theorem. Indeed, the Monotonicity Theorem rests upon the Mean-Value Theorem, the proof of which rests upon Rolle's Theorem.

**Exercise.** Modify the above proofs to prove the following fact. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Let $n \in \mathbb{Z}_+$. Suppose that the equation $f'(x) = 0$ has at most $n$ real solutions. Show that the equation $f(x) = 0$ has at most $n + 1$ real solutions.