First In-Class Exam
Math 410, Professor David Levermore
Thursday, 28 September 2017

1. [10] Let $a \in \mathbb{R}$ have the property that $a < 1/k$ for every $k \in \mathbb{Z}_+$. Prove $a \leq 0$.

2. [10] Prove that for every nonzero $x \in \mathbb{R}$ we have the inequality
$$1 + \tfrac{4}{3}x < (1+x)^{\frac{4}{3}} \,.$$

3. [15] Give a counterexample to each of the following false assertions.
   (a) If a sequence $\{a_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}$ diverges then the subsequence $\{a_{2k}\}_{k\in\mathbb{N}}$ diverges.
   (b) A countable intersection of nested nonempty open intervals is also nonempty.
   (c) If $\lim_{k\to\infty} a_k = 0$ then $\displaystyle\sum_{k=0}^{\infty} a_k$ converges.

4. [10] Consider the real sequence $\{b_k\}_{k\in\mathbb{N}}$ given by
$$b_k = (-1)^k \, \frac{2k+4}{k+1} \qquad \text{for every } k \in \mathbb{N} = \{0, 1, 2, \dots\} \,.$$
   (a) [3] Write down the first three terms of the subsequence $\{b_{2k}\}_{k\in\mathbb{N}}$.
   (b) [3] Write down the first three terms of the subsequence $\{b_{2^k}\}_{k\in\mathbb{N}}$.
   (c) [4] Write down $\liminf_{k\to\infty} b_k$ and $\limsup_{k\to\infty} b_k$. (No proof is needed here.)

5. [15] Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be bounded, positive sequences in $\mathbb{R}$.
   (a) [10] Prove that
$$\limsup_{k\to\infty} (a_k b_k) \leq \Big(\limsup_{k\to\infty} \, a_k\Big)\Big(\limsup_{k\to\infty} \, b_k\Big) \,.$$
   (b) [5] Give an example for which equality does not hold above.

6. [10] Let $\{a_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ be a sequence and $\{a_{n_k}\}_{k\in\mathbb{N}}$ be a subsequence of it. Show that
$$\sum_{k=0}^{\infty} a_k \quad \text{converges absolutely} \qquad \Longrightarrow \qquad \sum_{k=0}^{\infty} a_{n_k} \quad \text{converges absolutely} \,.$$

7. [10] Let $A \subset \mathbb{R}$ be bounded above. Let A^c denote the closure A. Prove $\sup\{A\} \in A^c$.

8. [10] Determine all $a \in \mathbb{R}$ for which
$$\sum_{k=0}^{\infty} \left(\frac{k^2+1}{k^4+1}\right)^a \quad \text{converges} \,.$$
   Give your reasoning.

9. [10] Let $\{b_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and let A be a subset of $\mathbb{R}$. Write the negations of the following assertions.
   (a) "For some $\epsilon > 0$ we have $|b_j - 3| \geq \epsilon$ frequently as $j \to \infty$."
   (b) "Every sequence in A has a subsequence that converges to a limit in A."