First In-Class Exam Solutions
Math 410, Professor David Levermore
Thursday, 28 September 2017

1. [10] Let $a \in \mathbb{R}$ have the property that $a < 1/k$ for every $k \in \mathbb{Z}_+$. Prove $a \leq 0$.

   Solution. This is a proof by contradiction. Suppose the negation of the assertion. Specifically, suppose that $a > 0$. Then by the Archimedean Property there exists an $n \in \mathbb{Z}_+$ such that $na > 1$, which implies that $\frac{1}{n} < a$. But this contradicts the property that $a < 1/k$ for every $k \in \mathbb{Z}_+$. Therefore $a \leq 0$. □

   Alternative Solution. The property that $a < 1/k$ for every $k \in \mathbb{Z}_+$ implies that a is a lower bound of the set $S = \{\frac{1}{k} : k \in \mathbb{Z}_+\}$. It follows that $a \leq \inf\{S\}$, because $\inf\{S\}$ is the greatest lower bound of the set S. We claim that $\inf\{S\} = 0$. Indeed, because $0 < \frac{1}{k}$ for every $k \in \mathbb{Z}_+$, we see that 0 is also a lower bound of S. Moreover, for every $b > 0$ the Archimedean Property implies there exists an $n \in \mathbb{Z}_+$ such that $nb > 1$. This implies that $\frac{1}{n} < b$, which means that b is not a lower bound for S. Therefore $\inf\{S\} = 0$, which implies that $a \leq 0$. □

2. [10] Prove that for every nonzero $x \in \mathbb{R}$ we have the inequality

$$1 + \tfrac{4}{3}x < (1+x)^{\frac{4}{3}}\,.$$

   Solution. Let $x \in \mathbb{R}$ be nonzero. Because for every $y, z \in \mathbb{R}$ we know that $y < z$ if and only if $y^{\frac{1}{3}} < z^{\frac{1}{3}}$, it is enough to prove the inequality

$$(1 + \tfrac{4}{3}x)^3 < (1+x)^4\,.$$

   By the binomial formula we have

$$\begin{aligned}(1+x)^4 &= 1 + 4x + 6x^2 + 4x^3 + x^4\,,\\ (1+\tfrac{4}{3}x)^3 &= 1 + 4x + \tfrac{16}{3}x^2 + \tfrac{64}{27}x^3\,.\end{aligned}$$

   Therefore

$$\begin{aligned}(1+x)^4 - (1+\tfrac{4}{3}x)^3 &= \big(6 - \tfrac{16}{3}\big)x^2 + \big(4 - \tfrac{64}{27}\big)x^3 + x^4\\ &= \tfrac{2}{3}x^2 + \tfrac{44}{27}x^3 + x^4 = x^2\big(x^2 + \tfrac{44}{27}x + \tfrac{2}{3}\big)\\ &= x^2\Big[\big(x - \tfrac{22}{27}\big)^2 + \tfrac{2}{3} - \big(\tfrac{22}{27}\big)^2\Big]\,.\end{aligned}$$

   Because $x \neq 0$ we see that $x^2 > 0$. Because

$$\tfrac{2}{3} - \big(\tfrac{22}{27}\big)^2 = \tfrac{2}{3} - \big(\tfrac{2}{3}\cdot\tfrac{11}{9}\big)^2 = \tfrac{2}{3}\big(1 - \tfrac{2}{3}\cdot\tfrac{121}{81}\big) = \tfrac{2}{3}\big(1 - \tfrac{242}{243}\big) = \tfrac{2}{3}\cdot\tfrac{1}{243} > 0\,,$$

   we see that the factor in the square brackets above is also positive. We conclude that for every nonzero $x \in \mathbb{R}$ we have

$$(1+\tfrac{4}{3}x)^3 < (1+x)^4\,,$$

   whereby for every nonzero $x \in \mathbb{R}$ we have

$$1 + \tfrac{4}{3}x < (1+x)^{\frac{4}{3}}\,.$$

   Therefore we have proved the asserted inequality. □

Alternative Solution. This is a proof by contradiction. Suppose the negation of the assertion. Specifically, suppose that there exists some nonzero $c \in \mathbb{R}$ such that

$$1 + \tfrac{4}{3}c \geq (1+c)^{\frac{4}{3}}\,.$$

Because for every $y, z \in \mathbb{R}$ we know that $z \geq y$ implies $z^3 \geq y^3$, it follows that

$$(1 + \tfrac{4}{3}c)^3 \geq (1+c)^4\,.$$

By the binomial formula we have

$$(1+c)^4 = 1 + 4c + 6c^2 + 4c^3 + c^4\,,$$
$$(1 + \tfrac{4}{3}c)^3 = 1 + 4c + \tfrac{16}{3}c^2 + \tfrac{64}{27}c^3\,.$$

Therefore the inequality satisfied by c becomes

$$\tfrac{16}{3}c^2 + \tfrac{64}{27}c^3 \geq 6c^2 + 4c^3 + c^4\,,$$

which implies that

$$\begin{aligned} 0 &\geq \left(6 - \tfrac{16}{3}\right)c^2 + \left(4 - \tfrac{64}{27}\right)c^3 + c^4 \\ &= \tfrac{2}{3}c^2 + \tfrac{44}{27}c^3 + c^4 = c^2\left(c^2 + \tfrac{44}{27}c + \tfrac{2}{3}\right) \\ &= c^2\left[\left(c - \tfrac{22}{27}\right)^2 + \tfrac{2}{3} - \left(\tfrac{22}{27}\right)^2\right]. \end{aligned}$$

Because $c \neq 0$ we see that $c^2 > 0$. Because

$$\tfrac{2}{3} - \left(\tfrac{22}{27}\right)^2 = \tfrac{2}{3} - \left(\tfrac{2}{3}\cdot\tfrac{11}{9}\right)^2 = \tfrac{2}{3}\left(1 - \tfrac{2}{3}\cdot\tfrac{121}{81}\right) = \tfrac{2}{3}\left(1 - \tfrac{242}{243}\right) = \tfrac{2}{3}\cdot\tfrac{1}{243} > 0\,,$$

we see that the factor in the square brackets above is also positive. Therefore we have shown that 0 is greater than or equal to a positive number, which is a contradiction. Hence, no such c exists. Therefore the asserted inequality holds. □

3. [15] Give a counterexample to each of the following false assertions.
   (a) If a sequence $\{a_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}$ diverges then the subsequence $\{a_{2k}\}_{k\in\mathbb{N}}$ diverges.
   (b) A countable intersection of nested nonempty open intervals is also nonempty.
   (c) If $\lim_{k\to\infty} a_k = 0$ then $\displaystyle\sum_{k=0}^{\infty} a_k$ converges.

Solution (a). A simple counterexample is obtained by setting $a_k = (-1)^k$ for every $k \in \mathbb{N}$. Then the sequence $\{a_k\}_{k\in\mathbb{N}}$ diverges but the subsequence $\{a_k^2\}_{k\in\mathbb{N}}$ converges to 1 because $a_k^2 = (-1)^{2k} = 1$. □

Solution (b). Any countable intersection of nested nonempty open intervals must have the form

$$\bigcap_{k=0}^{\infty}(a_k, b_k)$$

where $a_k < b_k$ and $(a_{k+1}, b_{k+1}) \subset (a_k, b_k)$ for every $k \in \mathbb{N}$. Such an intersection that is empty is obtained by setting $a_k = 0$ and $b_k = 2^{-k}$ for every $k \in \mathbb{N}$. □

Solution (c). A simple counterexample is obtained by setting $a_k = \frac{1}{k}$ for every $k \in \mathbb{Z}_+$ because

$$\lim_{k\to\infty}\frac{1}{k} = 0\,, \quad \text{while the harmonic series} \quad \sum_{k=1}^{\infty}\frac{1}{k} \quad \text{diverges}\,.$$

4. [10] Consider the real sequence $\{b_k\}_{k\in\mathbb{N}}$ given by

$$b_k = (-1)^k \, \frac{2k+4}{k+1} \qquad \text{for every } k \in \mathbb{N} = \{0, 1, 2, \dots\}\,.$$

(a) [3] Write down the first three terms of the subsequence $\{b_{2k}\}_{k\in\mathbb{N}}$.
(b) [3] Write down the first three terms of the subsequence $\{b_{2^k}\}_{k\in\mathbb{N}}$.
(c) [4] Write down $\liminf_{k\to\infty} b_k$ and $\limsup_{k\to\infty} b_k$. (No proof is needed here.)

Remark. We are given that $\mathbb{N} = \{0, 1, 2, \cdots\}$, as was the convention in the notes.

Solution (a). When $k = 0, 1, 2$ we have $2k = 0, 2, 4$, whereby the first three terms of the subsequence $\{b_{2k}\}_{k\in\mathbb{N}}$ are

$$b_0 = (-1)^0 \, \frac{2\cdot 0+4}{0+1} = 4\,, \quad b_2 = (-1)^2 \, \frac{2\cdot 2+4}{2+1} = \frac{8}{3}\,, \quad b_4 = (-1)^4 \, \frac{2\cdot 4+4}{4+1} = \frac{12}{5}\,.$$

Solution (b). When $k = 0, 1, 2$ we have $2^k = 1, 2, 4$, whereby the first three terms of the subsequence $\{b_{2^k}\}_{k\in\mathbb{N}}$ are

$$b_1 = (-1)^1 \, \frac{2\cdot 1+4}{1+1} = -3\,, \qquad b_2 = \frac{8}{3}\,, \qquad b_4 = \frac{12}{5}\,.$$

Solution (c) Because $b_{2k+1} < 0$ while $b_{2k} > 0$, and because

$$\lim_{k\to\infty} b_{2k+1} = \lim_{k\to\infty} \left((-1)^{2k+1} \frac{2(2k+1)+4}{(2k+1)+1} \right) = -\lim_{k\to\infty} \frac{4k+6}{2k+2} = -2\,,$$

while

$$\lim_{k\to\infty} b_{2k} = \lim_{k\to\infty} \left((-1)^{2k} \frac{2(2k)+5}{(2k)+2} \right) = \lim_{k\to\infty} \frac{4k+5}{2k+2} = 2\,,$$

we see that

$$\liminf_{k\to\infty} b_k = \lim_{k\to\infty} b_{2k+1} = -2\,, \qquad \limsup_{k\to\infty} b_k = \lim_{k\to\infty} b_{2k} = 2\,.$$

□

Remark. Proposition 2.17 can be used to prove the answers to (c).

5. [15] Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be bounded, positive sequences in $\mathbb{R}$.
(a) [10] Prove that

$$\limsup_{k\to\infty} (a_k b_k) \le \Big(\limsup_{k\to\infty} \, a_k \Big) \Big(\limsup_{k\to\infty} \, b_k \Big)\,.$$

(b) [5] Give an example for which equality does not hold above.

Remark. This is a variant of the problem from the homework to prove that

$$\limsup_{k\to\infty} (a_k + b_k) \le \limsup_{k\to\infty} \, a_k + \limsup_{k\to\infty} \, b_k\,.$$

The proofs of these two inequalities are very similar.

Solution (a). Let $c_k = a_k b_k$ for every $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ we define

$$\overline{a}_k = \sup\{a_l \,:\, l \geq k\}\,, \qquad \overline{b}_k = \sup\{b_l \,:\, l \geq k\}\,, \qquad \overline{c}_k = \sup\{c_l \,:\, l \geq k\}\,.$$

Because the sequences $\{a_k\}_{k\in\mathbb{N}}$, $\{b_k\}_{k\in\mathbb{N}}$, and $\{c_k\}_{k\in\mathbb{N}}$ are bounded above and positive, for every $k \in \mathbb{N}$ we have

$$0 < \overline{a}_k < \infty\,, \qquad 0 < \overline{b}_k < \infty\,, \qquad 0 < \overline{c}_k < \infty\,.$$

The real sequences $\{\overline{a}_k\}_{k\in\mathbb{N}}$, $\{\overline{b}_k\}_{k\in\mathbb{N}}$, and $\{\overline{c}_k\}_{k\in\mathbb{N}}$ are nonincreasing because their terms are supremums of successively smaller sets. Moreover, they are bounded below because $\{a_k\}_{k\in\mathbb{N}}$, $\{b_k\}_{k\in\mathbb{N}}$, and $\{c_k\}_{k\in\mathbb{N}}$ are positive. Therefore they converge by Montonic Sequence Convergence Theorem. By the definition of lim sup we have

$$\limsup_{k\to\infty} a_k = \lim_{k\to\infty} \overline{a}_k\,, \qquad \limsup_{k\to\infty} b_k = \lim_{k\to\infty} \overline{b}_k\,, \qquad \limsup_{k\to\infty} c_k = \lim_{k\to\infty} \overline{c}_k\,.$$

The crucial observation is that for every $k \in \mathbb{N}$ we have

$$c_l = a_l b_l \leq \overline{a}_k \overline{b}_k \quad \text{for every } l \geq k\,,$$

which yields the inequality

$$\overline{c}_k = \sup\{c_l \,:\, l \geq k\} \leq \overline{a}_k \overline{b}_k\,.$$

This inequality and the properties of limits then imply

$$\begin{aligned}
\limsup_{k\to\infty} a_k b_k = \limsup_{k\to\infty} c_k = \lim_{k\to\infty} \overline{c}_k \leq \lim_{k\to\infty} \overline{a}_k \overline{b}_k \\
&= \Big(\lim_{k\to\infty} \overline{a}_k\Big)\Big(\lim_{k\to\infty} \overline{b}_k\Big) \\
&= \Big(\limsup_{k\to\infty} a_k\Big)\Big(\limsup_{k\to\infty} b_k\Big)\,.
\end{aligned}$$

This is the inequality that we were asked to prove. □

Solution (b). Let $\rho > 1$. Let $\{a_k\}_{k\in\mathbb{N}}$ be any bounded, positive sequence such that

$$\liminf_{k\to\infty} a_k = \frac{1}{\rho}\,, \qquad \text{and} \qquad \limsup_{k\to\infty} a_k = \rho\,.$$

For example, we can simply take

$$a_k = \rho^{(-1)^k} = \begin{cases} \rho & \text{for } k \text{ even} \\ \dfrac{1}{\rho} & \text{for } k \text{ odd}\,. \end{cases}$$

Set $b_k = 1/a_k$ for every $k \in \mathbb{N}$. Then

$$\limsup_{k\to\infty} b_k = \limsup_{k\to\infty} \frac{1}{a_k} = \frac{1}{\liminf_{k\to\infty} a_k} = \rho\,,$$

whereby

$$\limsup_{k\to\infty} a_k b_k = 1 < \rho^2 = \Big(\limsup_{k\to\infty} a_k\Big)\Big(\limsup_{k\to\infty} b_k\Big)\,.$$

So equality does not hold for the bounded, positive sequences $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$. □

6. [10] Let $\{a_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ be a sequence and $\{a_{n_k}\}_{k\in\mathbb{N}}$ be a subsequence of it. Show that

$$\sum_{k=0}^{\infty} a_k \quad \text{converges absolutely} \qquad \Longrightarrow \qquad \sum_{k=0}^{\infty} a_{n_k} \quad \text{converges absolutely}\,.$$

Solution. By the definition of absolute convergence of a series

$$\sum_{k=0}^{\infty} a_k \quad \text{converges absolutely} \qquad \Longleftrightarrow \qquad \sum_{k=0}^{\infty} |a_k| \quad \text{converges}\,,$$

$$\sum_{k=0}^{\infty} a_{n_k} \quad \text{converges absolutely} \qquad \Longleftrightarrow \qquad \sum_{k=0}^{\infty} |a_{n_k}| \quad \text{converges}\,.$$

By the definition of a convergent series, each of the series on the right-hand side above is convergent if and only if its associated sequence of partial sums is convergent. These sequences of partial sums are given by $\{q_n\}$ and $\{p_m\}$ respectively where q_n and p_m are defined for every $n, m \in \mathbb{N}$ by

$$q_n = \sum_{k=0}^{n} |a_k|\,, \qquad\qquad p_m = \sum_{k=0}^{m} |a_{n_k}|\,.$$

It is clear that these sequences are nondecreasing. The Monotonic Sequence Theorem then implies that these sequences converge if and only if they are bounded above. Therefore

$$\sum_{k=0}^{\infty} |a_k| \quad \text{converges} \qquad \Longleftrightarrow \qquad \{q_n\} \text{ is bounded above}\,,$$

$$\sum_{k=0}^{\infty} |a_{n_k}| \quad \text{converges} \qquad \Longleftrightarrow \qquad \{p_m\} \text{ is bounded above}\,.$$

The crucial observation is that p_m and q_n satisfy the inequality

$$p_m = \sum_{k=0}^{m} |a_{n_k}| \leq \sum_{k=0}^{n_m} |a_k| = q_{n_m} \quad \text{for every } m \in \mathbb{N}\,.$$

This inequality shows that if $\{q_n\}$ is bounded above then $\{p_m\}$ is bounded above. Therefore

$$\begin{aligned}
\sum_{k=0}^{\infty} a_k \quad \text{converges absolutely} \qquad &\Longleftrightarrow \qquad \{q_n\} \text{ is bounded above} \\
&\Longrightarrow \qquad \{p_m\} \text{ is bounded above} \\
&\Longleftrightarrow \qquad \sum_{k=0}^{\infty} a_{n_k} \quad \text{converges absolutely}\,.
\end{aligned}$$

□

Remark. This proof involves three notions of convergence: (1) absolute convergence of a series, (2) convergence of a series, and (3) convergence of a sequence. Whenever "converges" appears in your solution it should be clear which notion is being used.

7. [10] Let $A \subset \mathbb{R}$ be bounded above. Let A^c denote the closure A. Prove $\sup\{A\} \in A^c$.

Solution. Because $A \subset \mathbb{R}$ is bounded above, we know A has an upper bound in $\mathbb{R}$. Therefore $\sup\{A\}$, which is the least upper bound of A, is also in $\mathbb{R}$. Let $a = \sup\{A\}$. We want to show that $a \in A^c$.

For every $n \in \mathbb{N}$ the set $A \cap (a - 2^{-n}, \infty)$ is nonempty because otherwise every point in $(a - 2^{-n}, \infty)$ would be an upper bound of A, which contradicts the fact that a is the least upper bound of A. Now for every $n \in \mathbb{N}$ let $a_n \in A \cap (a - 2^{-n}, \infty)$. Then because a is an upper bound for A we have

$$a - 2^{-n} < a_n \le a \quad \text{for every } n \in \mathbb{N}\,.$$

Therefore

$$|a_n - a| = a - a_n < 2^{-n} \quad \text{for every } n \in \mathbb{N}\,,$$

which implies that $a_n \to a$ as $n \to \infty$. Because $\{a_n\}_{n\in\mathbb{N}} \subset A$ and $a_n \to a$ as $n \to \infty$, we conclude that $a \in A^c$. □

8. [10] Determine all $a \in \mathbb{R}$ for which

$$\sum_{k=0}^{\infty} \left(\frac{k^2+1}{k^4+1} \right)^a \quad \text{converges}\,.$$

Give your reasoning.

Solution. The series converges for $a \in (\frac{1}{2}, \infty)$ and diverges otherwise. Because

$$\frac{k^2+1}{k^4+1} \sim \frac{1}{k^2} \quad \text{as } k \to \infty\,,$$

we see that the original series should be compared with the p-series

$$\sum_{k=1}^{\infty} \frac{1}{k^{2a}}\,.$$

This is best handled by Two-Way Limit Comparison Test. Indeed, for every $a \in \mathbb{R}$ we have

$$\lim_{k\to\infty} \frac{\left(\dfrac{k^2+1}{k^4+1} \right)^a}{\dfrac{1}{k^{2a}}} = \lim_{k\to\infty} \left(\frac{k^4+k^2}{k^4+1} \right)^a = \lim_{k\to\infty} \left(\frac{1+\dfrac{1}{k^2}}{1+\dfrac{1}{k^4}} \right)^a = 1\,,$$

so the Two-Way Limit Comparison Test implies that

$$\sum_{k=0}^{\infty} \left(\frac{k^2+1}{k^4+1} \right)^a \quad \text{converges} \quad \Longleftrightarrow \quad \sum_{k=1}^{\infty} \frac{1}{k^{2a}} \quad \text{converges}\,.$$

Because the $p = 2a$ for the p-series, it converges for $a \in (\frac{1}{2}, \infty)$ and diverges otherwise. The same is thereby true for the original series. □

Remark. The Two-Way Direct Comparison Test can be used, but not as efficiently because the sign of a matters. A proof might start with the bounds

$$\frac{1}{k^2} \le \frac{k^2+1}{k^4+1} \le \frac{2}{k^2} \quad \text{for every } k \in \mathbb{Z}_+\,.$$

9. [10] Let $\{b_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and let A be a subset of $\mathbb{R}$. Write the negations of the following assertions.
   (a) "For some $\epsilon > 0$ we have $|b_j - 3| \geq \epsilon$ frequently as $j \to \infty$."
   (b) "Every sequence in A has a subsequence that converges to a limit in A."

   Solution (a). "For every $\epsilon > 0$ we have $|b_j - 3| < \epsilon$ eventually as $j \to \infty$." □

   Solution (b). "There is a sequence in A such that every subsequence of it either diverges or converges to a limit outside A." □

   Remark. The answer "There is a sequence in A such that no subsequence of it converges to a limit in A." does not fully carry the negation through.

   Remark. Assertion (a) is equivalent to the sequence $\{b_k\}$ converges to 3. Assertion (b) is the definition that the set A is sequentially compact.