MATH 410, Assignment #1, due February 7

1. Use induction to prove the following: For $n = 2, 3, 4, \dots$ and $x, p \in \mathbb{R}$

$$x^n = p^n + np^{n-1}(x-p) + (x-p)^2 \sum_{k=1}^{n-1} kp^{k-1}x^{n-1-k} \tag{1}$$

Hint: First write down (1) for $n = 4$. Multiply both sides by x and try to obtain (1) for $n = 5$. This should help you understand how to do the inductive step from n to $n+1$.

2. Prove the following statement $(*)$: For $b \in \mathbb{R}_+$ and $n \in \mathbb{N}$ there exists a unique $a \in \mathbb{R}_+$ with $a^n = b$. Proceed as follows:

(a) Show: For $p \in \mathbb{R}_+$ with $p^n > b$ there exists $q < p$ with $q^n > b$. *Hint:* Define $f(x) := x^n - b$ and the tangent line $g(x) := f(p) + f'(p)(x-p)$. Define q such that $g(q) = 0$. Show that $0 < q < p$. Show that $g(q) < f(q)$ using (1).

(b) Show: For $\tilde{p} \in \mathbb{R}_+$ with $\tilde{p}^n < b$ there exists $\tilde{q} > \tilde{p}$ with $\tilde{q}^n < b$. *Hint:* Use $p = b/\tilde{p}^{n-1}$ and $q = b/\tilde{q}^{n-1}$ and (a). You can use that $c^{1/(n-1)}$ is already defined (statement $(*)$ for $n-1$).

(c) Now prove the statement $(*)$. See the proof for $b^{1/2}$ on the web page.

3. We are given $b \in \mathbb{R}_+$ and $n \in \{2, 3, 4, \dots\}$. We want to compute $b^{1/n}$ using a calculator which can only add, subtract, multiply, divide. We use an initial guess a_0 with $a_0^n > b$ (e.g., $a_0 = b + 1$). We compute for $k = 0, 1, 2, \dots$

$$a_{k+1} := a_k - \frac{f(a_k)}{f'(a_k)}$$

with $f(x) := x^n - b$.

(a) Show: The sequence a_k with $k \in \mathbb{N}_0$ is decreasing and $a_k > b^{1/n}$.

(b) Show: The sequence a_k converges. Let $a_* := \lim_{k\to\infty} a_k$. Show that $a_* = b^{1/n}$. *Hint:* Use $a_{k+1} = a_k - \dfrac{a_k^n - b}{na_k^{n-1}}$ and take the limit for $k \to \infty$ on both sides of this equation.

(c) Show that the error $a_k - a_* > 0$ converges very quickly to 0:

$$a_{k+1} - a_* \le \frac{n-1}{2a_*}(a_k - a_*)^2 \qquad \text{for } k = 0, 1, 2, \dots$$

This means that the size of the error decreases e.g. from 10^{-4} to 10^{-8} to 10^{-16}, i.e., the number of correct digits doubles with each step.
Hint: (1) with $p = a_k$ and $x = a_*$ gives $f(a_*) = g(a_*) + (a_* - a_k)^2 S$ where S denotes the sum in (1). Use $f(a_*) = 0 = g(a_{k+1})$ to obtain $g(a_{k+1}) - g(a_*) = (a_* - a_k)^2 S$. This gives

$$a_{k+1} - a_* = (a_* - a_k)^2 \frac{S}{f'(a_k)}$$

Then use $a_* < a_k$ to show $S < (1 + 2 + \cdots + n - 1)a_k^{n-2}$.