MATH410: Solution Assignment 1

Problem 1:

Show that statement A_n holds for $n = 2, 3, \ldots$

$$\text{statement } A_n: \qquad x^n = p^n + np^{n-1}(x-p) + (x-p)^2 \sum_{k=1}^{n-1} kp^{k-1}x^{n-1-k} \tag{1}$$

Show: statment A_2 holds: We have to show

$$x^2 = p^2 + 2p(x-p) + (x-p)^2$$

We obtain this by applying the distributive property on the right hand side.

Show: statement A_n implies statement A_{n+1} for $n = 2, 3, \ldots$: Multiplying A_n by x and using $x = (x-p)+p$ gives

$$x^{n+1} = p^n x + \underbrace{np^{n-1}(x-p)x}_{np^{n-1}(x-p)^2 + \underbrace{np^{n-1}(x-p)p}_{np^n x \quad \underbrace{-np^{n+1}}_{-(n+1)p^{n+1}+p^{n+1}}}} + (x-p)^2\left[1\cdot p^0x^{n-2} + 2p^1x^{n-3} + \cdots + (n-1)p^{n-2}x^0\right]x$$

$$x^{n+1} = p^{n+1} + (n+1)p^n x - (n+1)p^n p + (x-p)^2\left[1\cdot p^0x^{n-1} + 2p^1x^{n-2} + \cdots + (n-1)p^{n-2}x^1 + np^{n-1}x^0\right]$$

which is statement A_{n+1}.

Problem 2(a):

Let $f(x) := x^n - b$, $g(x) := f(p) + f'(p)(x-p)$. Assume $p \in \mathbb{R}_+$ with $p^n > b$. Define $q := p - \dfrac{p^n - b}{np^{n-1}}$.

Show $q < p$: Since $p^n > b$, $p > 0$, we have $\frac{p^n-b}{np^{n-1}} > 0$.

Show $0 < q$: Since $b > 0$, we have $q = p - \dfrac{p^n - b}{np^{n-1}} > p - \dfrac{p^n}{np^{n-1}} = (1 - \dfrac{1}{n})p > 0$ for $n \geq 2$.

Show $g(q) < f(q)$: From (1) we get with $S := \sum_{k=1}^{n-1} kp^{k-1}x^{n-1-k}$

$$\begin{aligned} x^n - b &= p^n - b + np^{n-1}(x-p) + (x-p)^2 S \\ f(x) &= g(x) + (x-p)^2 S \\ f(q) &= g(q) + \underbrace{(q-p)^2}_{>0} \sum_{k=1}^{n-1} \underbrace{kp^{k-1}q^{n-1-k}}_{>0} \end{aligned} \tag{2}$$

Problem 2(b):

Assume $\tilde{p}^n > b$. Let $p := b/\tilde{p}^{n-1}$. Then $p^n = b^n/(\tilde{p}^n)^{n-1} > b^n/b^{n-1} = b$. From (a) we obtain $q < p$ with $q^n > b$.
Now we can define $\tilde{q} := (b/q)^{1/(n-1)}$: We have $c := b/q > 0$. Using induction we can assume that $(*)$ holds for $n-1$, hence there is a unique $d = c^{1/(n-1)} \in \mathbb{R}_+$ with $d^{n-1} = c$).
We have $b/q > b/p$. Hence (3) gives $\tilde{q} := (b/q)^{1/(n-1)} > (b/p)^{1/(n-1)} = \tilde{p}$. From $q^n > b$ we get $\tilde{q}^n = b^n/(q^n)^{(n-1)} < b^n/b^{n-1} = b$.

Problem 2(c):

Show: For $b \in \mathbb{R}_+$ and $n \in \{2,3,4,\dots\}$ there exists a unique $a \in \mathbb{R}_+$ with $a^n = b$.
We will use: For $y,z \in \mathbb{R}_+$ and $n \in \{2,3,4,\dots\}$ we have

$$y^n > z^n \implies y > z \tag{3}$$

Proof: Assume $y \le z$. Then Proposition 1.10(b) in the notes implies $y^n \le z^n$.

Proof of existence: Consider the set $A := \{x \in \mathbb{R}_+ : x^n < b\}$.
The set A is nonempty: We have $s := b/(b+1) < 1$, hence $s^n < s$. We also have $s < b$, hence $s^n < s < b$, i.e., $s \in A$.
The set A is bounded from above: We have $t := b+1 > 1$, hence $t^n > t > b$. Let $x \in A$, then $x^n < b < t^n$ which implies $x < t$ by (3). Therefore t is an upper bound for the set A.
By the least upper bound property the set A has a least upper bound

$$a := \sup A \tag{4}$$

We now show that $a^n = b$ using trichotomy:
Assume $a^n < b$ holds. Then by (b) there exists $\tilde{q} > a$ with $\tilde{q}^n < b$, i.e., $\tilde{q} \in A$. Then a is not an upper bound of A, contradicting (4).
Assume $a^n > b$ holds. Then by (a) there exists $q < a$ with $q^n > b$. Let $x \in A$, then $x^n < b < q^n$ which implies $x < q$ by (3). Therefore q is an upper bound for the set A. But $q < a$ contradicts that a is the least upper bound.
Proof of uniqueness: This follows from (3).

Problem 3(a):

We have $a_0^n > b$. Assume $a_k^n > b$ for $k \in \{0,1,2,\dots\}$. Then we obtain from 2(a) that $a_{k+1} = a_k - \frac{a_k^n - b}{n a_k^{n-1}}$ satisfies $a_{k+1} < a_k$ and $a_{k+1}^n - b > 0$. Therefore we obtain by induction that the statements $a_{k+1} < a_k$ and $a_k^n > b$ hold for all $k \in \{0,1,2,\dots\}$.

Problem 3(b):

The sequence a_k is nonincreasing and bounded from below. By the monotonic sequence theorem the sequence a_k converges. Hence the limit $a_* := \lim_{k\to\infty} a_k$ exists. $a_k > b^{1/n}$ implies $a_* \ge b^{1/n} > 0$. In $a_{k+1} = a_k - \dfrac{a_k^n - b}{n a_k^{n-1}}$ we take the limit for $k \to \infty$, yielding with Proposition 2.9 $a_* = a_* - \dfrac{a_*^n - b}{n a_*^{n-1}}$. Hence $a_* > 0$ satisfies $a_*^n = b$. By problem 2 we must have $a_* = b^{1/n}$.

Problem 3(c):

Let $p = a_k$ and $x = a_*$. Then (2) gives $f(a_*) = g(a_*) + (a_* - a_k)^2 S$ with $S = \sum_{j=1}^{n-1} j a_k^{j-1} a_*^{n-1-j}$. We have $f(a_*) = a_*^n - b = 0$. Note that a_{k+1} is defined so that $g(a_{k+1}) = 0$. We get

$$\begin{aligned} f(a_*) &= g(a_*) + (a_* - a_k)^2 S \\ \underbrace{g(a_{k+1}) - g(a_*)}_{f'(a_k)(a_{k+1} - a_*)} &= (a_* - a_k)^2 S \\ a_{k+1} - a_* &= (a_* - a_k)^2 \frac{S}{n a_k^{n-1}} \end{aligned}$$

From (a) we have $a_* < a_k$, hence $S = \sum_{j=1}^{n-1} j a_k^{j-1} a_*^{n-1-j} < \sum_{j=1}^{n-1} j a_k^{n-2} = \frac{n(n-1)}{2} a_k^{n-2}$ since $\sum_{j=1}^{n-1} j = \frac{n(n-1)}{2}$ (this was proved in class). This gives

$$a_{k+1} - a_* \le (a_* - a_k)^2 \frac{\frac{n(n-1)}{2} a_k^{n-2}}{n a_k^{n-1}} = (a_* - a_k)^2 \frac{n-1}{2a_k} \le (a_* - a_k)^2 \frac{n-1}{2a_*}$$