Math 410: Solution of Exam 1

1. Let $A = \{\frac{p}{q} : p, q \in \mathbb{N} \text{ with } p < q\}$. Find $\inf A$, $\sup A$ and $\mathrm{cl}(A)$ and justify your answers.
A is bounded from below by 0. A contains the sequence $\frac{1}{n}$ which converges to 0. Hence $0 = \inf A$.
A is bounded from above by 1. A contains the sequence $\frac{n-1}{n}$ which converges to 1. Hence $1 = \sup A$.
claim: $\mathrm{cl}(A) = [0, 1]$. We already found sequences which converge to 0 and 1. Let $a \in (0, 1)$ and $k \in \mathbb{N}$. Then for $x := ka$ there exists $n_k \in \mathbb{N}_0$ with $n_k < ka \leq n_k + 1$, hence $a \in (\frac{n_k}{k}, \frac{n_k+1}{n}]$. Hence the sequence $\frac{n_k}{k} \in A$ converges to a.

2. Find $\liminf_{k\to\infty} a_k$ and $\limsup_{k\to\infty} a_k$ for the following sequences with $k \in \mathbb{N}$:

(a) $a_k = (-k)^k + k^k + \frac{1}{k}$
For even k we have $a_k = 2k^k + \frac{1}{k}$ which diverges to $+\infty$ as $k \to \infty$.
For odd k we have $a_k = \frac{1}{k}$ which converges to 0 as $k \to \infty$. Hence $\liminf_{k\to\infty} a_k = 0$ and $\limsup_{k\to\infty} a_k = \infty$.

(b) $a_k = \left(-1 - \frac{1}{k}\right)^k$
For even k we have $a_k = (1 + \frac{1}{k})^k$ which converges to e as $k \to \infty$.
For odd k we have $a_k = -(1 + \frac{1}{k})^k$ which converges to $-e$ as $k \to \infty$. Hence $\liminf_{k\to\infty} a_k = -e$ and $\limsup_{k\to\infty} a_k = e$.

(c) $a_k = (k+1)^{1/3} - k^{1/3}$. *Hint:* Use $x^n - y^n = (x-y)(x^{n-1}y^0 + x^{n-2}y + \cdots + x^0y^{n-1})$ with $x = (k+1)^{1/3}$, $y = k^{1/3}$
With $n = 3$ we get $x - y = \dfrac{x^3 - y^3}{x^2 + xy + y^2}$ which gives

$$a_k = (k+1)^{1/3} - k^{1/3} = \frac{1}{(k+1)^{2/3} + (k+1)^{1/3}k^{1/3} + k^{2/3}}$$

Since the denominator tends to ∞ as $k \to \infty$ we have that $a_k \to 0$ as $k \to \infty$. Hence $\liminf_{k\to\infty} a_k = \limsup_{k\to\infty} a_k = 0$.

3. Find out for which $x \in \mathbb{R}$ the series converges, and for which $x \in \mathbb{R}$ the series diverges. Explain your reasoning. Be careful to discuss all possible cases for x, including the "borderline" cases.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k^{1/2}2^k}$
We use the ratio test: $\dfrac{|a_{k+1}|}{|a_k|} = \left(\dfrac{k}{k+1}\right)^{1/2} \dfrac{|x|}{2} = \left(1 + \dfrac{1}{k}\right)^{1/2} \dfrac{|x|}{2} \to \dfrac{|x|}{2} =: \rho$. Now the ratio test gives:
(i) The series converges for $\rho < 1 \iff |x| < 2$.
(ii) The series diverges for $\rho > 1 \iff |x| > 2$.
It remains to investigate the borderline cases $x = 2$ and $x = -2$.
For $x = 2$ we obtain the series $\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$ which diverges to ∞. (Recall: $\sum_{k=1}^{\infty} k^p$ converges for $p < -1$ and diverges for $p \geq -1$.)
For $x = -2$ we obtain the series $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^{1/2}}$ which is an alternating series with $a_k \to 0$, hence it converges.

(b) $\displaystyle\sum_{k=1}^{\infty} (1 + k^4)^x$. We compare $a_k := (1 + k^4)^x > 0$ with $b_k := k^{4x} > 0$. Then $\dfrac{a_k}{b_k} = \left(\dfrac{1 + k^4}{k^4}\right)^x = \left(k^{-4} + 1\right)^x \to 1$ as $k \to \infty$. For $q \in (0, 1)$ we have therefore $qb_k \leq a_k \leq q^{-1}b_k$ eventually. This gives:
(i) For $4x < -1 \iff x < -\frac{1}{4}$ the series $\sum b_k$ converges, hence $\sum a_k$ converges
(ii) For $4x \geq -1 \iff x \geq -\frac{1}{4}$ the series $\sum b_k$ diverges to ∞, hence $\sum a_k$ diverges to ∞.

(c) $\sum_{k=1}^{\infty} \frac{(2k)!}{k^{2k}} x^k$

We use the ratio test: $\frac{|a_{k+1}|}{|a_k|} = \frac{(2k+2)(2k+1)}{k^2} |x| = \left(2 + \frac{2}{k}\right)\left(2 + \frac{1}{k}\right) |x| \to 4|x| =: \rho$ as $k \to \infty$. Now the ratio test gives:
(i) The series converges for $\rho < 1 \iff |x| < \frac{1}{4}$
(ii) The series diverges for $\rho > 1 \iff |x| > \frac{1}{4}$.
It remains to investigate the borderline cases $x = \frac{1}{4}$ and $x = -\frac{1}{4}$. In this case we have $|a_k| = \frac{(2k)!}{k^{2k}4^k} = \frac{(2k)!}{(2k)^{2k}} = \prod_{j=1}^{2k} \frac{j}{k} \leq \left(\frac{1}{2}\right)^k 1^k$ since the first k factors are bounded by $\frac{1}{2}$ and the remaining factors are bounded by 1. Hence $\sum |a_k|$ can be bounded from above by a convergent geometric series. Therefore the series $\sum a_k$ converges absolutely for $|x| = \frac{1}{4}$.

4. For the following statements give either a proof or a counterexample.

(a) For every sequence $a_k \in \mathbb{R}$ there exists $a \in \mathbb{R} \cup \{\infty, -\infty\}$ and a subsequence a_{n_k} with $a_{n_k} \to a$ as $k \to \infty$.
Proof: **Case 1:** the sequence a_k is unbounded. If it is not bounded above there exists a subsequence $a_{n_k} \to \infty$. If it is not bounded below there exists a subsequence $a_{n_k} \to -\infty$.
Case 2: The sequence a_k is bounded. Now the Bolzano-Weierstrass theorem states that there exists a subsequence which converges to a limit $a \in \mathbb{R}$.

(b) If $\sum_{k=1}^{\infty} a_k$ converges, then for any subsequence a_{n_k} the series $\sum_{k=1}^{\infty} a_{n_k}$ converges.
Counterexample: $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ converges (alternating series with $a_k \to 0$). However, for the subsequence $n_k = 2k$ we get the series $\sum_{k=1}^{\infty} a_{n_k} = \sum_{k=1}^{\infty} \frac{1}{2k} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k}$ which diverges to ∞.

(c) If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then for any subsequence a_{n_k} the series $\sum_{k=1}^{\infty} a_{n_k}$ converges.
Proof: We have that $\sum_{k=1}^{\infty} |a_k| =: S < \infty$ converges. The sequence $s_k := \sum_{j=1}^{k} |a_{n_j}|$ is nondecreasing and bounded from above by S, hence s_k converges. This means that the series $\sum_{k=1}^{\infty} |a_{n_k}|$ is absolutely convergent, hence it is convergent.