Math 410: Solution of Exam 2

1. Let $f(x) = e^{-2x}$. We want to approximate $y := f(0.1)$.

(a) Find the Taylor polynomial $p_2(x) = f(a) + \cdots + (\cdots)(x-a)^2$ about the point $a = 0$. Find $\tilde{y} := p_2(0.1)$. Simplify your answers.
$f(0) = 1$, $f'(x) = -2e^{-2x}$, $f'(0) = -2$, $f''(x) = 4e^{-2x}$, $f''(0) = 4$,
$p_2(x) = f(0) + f'(0)(x-0) + \frac{1}{2}f''(0)(x-0)^2 = \boxed{1 - 2x + 2x^2}$, $\tilde{y} = p_2(0.1) = 1 - .2 + .02 = \boxed{.82}$

(b) Write down the Lagrange remainder term $y - \tilde{y} = \cdots$. Find an upper bound $|y - \tilde{y}| \le \cdots$, give a number like e.g. $10^{-5}/3$.
$y - \tilde{y} = R_2 = \frac{1}{3!}f'''(p)x^3$ with $x = .1$ and $p \in (0, .1)$. Here $f'''(p) = -8e^{-2x}$. Since $e^{-p} \le 1$ we obtain $\boxed{y - \tilde{y} = R_2 = \frac{1}{6}(-8)e^{-2p}(.1)^3}$ and $|y - \tilde{y}| = \left|\frac{1}{6}(-8)e^{-2p}(.1)^3\right| = \frac{4}{3}e^{-2p}(.1)^3 \le \boxed{\frac{4}{3}\cdot 10^{-3}}$.

2. Determine the following **1-sided limits** for $x \to 0^-$: Specify (i) a number $L \in \mathbb{R}$, (ii) $+\infty$ or $-\infty$, (iii) none of the above. Justify your answers.

(a) $\lim_{x\to 0^-} \frac{e^{-x} - 1}{x^2}$: Plugging in $x = 0$ gives $\frac{0}{0}$, L'Hospital gives $\lim_{x\to 0^-} \frac{-e^{-x}}{2x}$ which gives $\boxed{+\infty}$ (numerator$\to 1$, denominator$\to 0$ from below)

(b) $\lim_{x\to 0^-} \frac{e^{-x} - 1 + x}{x^2}$: Plugging in $x = 0$ gives $\frac{0}{0}$, L'Hospital gives $\lim_{x\to 0^-} \frac{-e^{-x} + 1}{2x}$. Plugging in $x = 0$ gives $\frac{0}{0}$, using L'Hospital again gives $\lim_{x\to 0^-} \frac{e^{-x}}{2} = \boxed{\frac{1}{2}}$

3. Let $D = [0, 3]$. We have a **differentiable** function $f\colon D \to \mathbb{R}$ with the following function values:

x	0	1	2	3
$f(x)$	1	2	-1	1

(a) Show: there are two *different* points $x \in D$ with $f(x) = 0$.
f differentiable $\implies$ f continuous. **Intermediate value theorem:** (i) exists $p_1 \in (1, 2)$ with $f(p_1) = 0$, (ii) exists $p_2 \in (2, 3)$ with $f(p_2) = 0$.

(b) Show: there exists $x \in D$ with $f'(x) > 1$, there exists $x \in D$ with $f'(x) < -1$.
Show: there exist two *different* points $x \in D$ with $f'(x) = 0$.
Mean value theorem: $\exists s_1 \in (1, 2)$ with $f'(s_1) = \frac{f(2) - f(1)}{2 - 1} = \frac{-3}{1} < -1$, $\exists s_2 \in (2, 3)$ with $f'(s_2) = \frac{f(3) - f(2)}{3 - 2} = \frac{2}{1} > 1$,
$\exists s_0 \in (0, 1)$ with $f'(s_0) = \frac{2-1}{1-0} > 0$
Derivative intermediate value theorem: $\exists q_1 \in (s_0, s_1)$ with $f'(q_1) = 0$, $\exists q_2 \in (s_1, s_2)$ with $f'(q_2) = 0$

(c) Assume f is twice differentiable on D. Show: there exists $x \in D$ with $f''(x) = 0$.
Use (b) and **mean value theorem for f':** $\exists p \in (q_1, q_2)$ with $f''(p) = \frac{f'(q_2) - f'(q_1)}{q_2 - q_1} = 0$ (or **Rolle's theorem**)

4. Are the following functions $f\colon D \to \mathbb{R}$ (i) Cauchy continuous, (ii) uniformly continuous? Use the results from class to justify your answers.

(a) $D = (0, 1)$, $f(x) = x^{-2} + x$: For Cauchy sequence $x_n = \frac{1}{n}$ we have $f(x_n) = n^2 + n \to \infty$ as $n \to \infty$, i.e., $f(x_n)$ is NOT a Cauchy sequence. Hence f is NOT Cauchy continuous on D. Therefore f is NOT uniformly continuous on D (since unif. contin. $\implies$ Cauchy contin.).

(b) $D = (1, \infty)$, $f(x) = x^{-2} + x$: $f'(x) = -2x^{-3} + 1$ is bounded on $D = (1, \infty)$, hence f is Lipschitz on D. Therefore f is uniformly continuous on D, and hence also Cauchy continuous on D.