Math 410: Solution Final Exam Spring 2018

1. (20 pts) Find out for which $x \in \mathbb{R}$ the series converges, and for which $x \in \mathbb{R}$ the series diverges. Explain your reasoning. Be careful to discuss all possible cases for x, including the "borderline" cases.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k4^k}$

Ratio test: $\dfrac{a_{k+1}}{a_k} = \dfrac{k}{k+1} \cdot \dfrac{x}{4}$, $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{x}{4}\right|$, hence: $|x| < 4 \implies$ convergence, $|x| > 4 \implies$ divergence

borderline cases: for $x = 4$ we have $\sum_{k=1}^{\infty} \frac{1}{k}$ which diverges, for $x = -4$ we have $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ which converges (alternating series with $a_k \to 0$).

(b) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k^2}{2+k^3}\right)^x$

Use comparison: $\alpha_k := \frac{k^2}{2+k^3}$ behaves like $\beta_k := \frac{1}{k}$ for large k, $\frac{\alpha_k}{\beta_k} \to 1$ as $k \to \infty$

Hence for $a_k := \left(\frac{k^2}{2+k^3}\right)^x$ and $b_k = \frac{1}{k^x}$ we have $\frac{a_k}{b_k} = \left(\frac{\alpha_k}{\beta_k}\right)^x \to 1$ as $k \to \infty$

Two-way comparison test for series with nonnegative terms: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^x}$ converges for $x > 1$, diverges for $x \le 1$, therefore the same holds for $\sum_{k=1}^{\infty} a_k$.

2. (20 pts) Prove the following statements:

(a) Let x_k be a bounded sequence which is not convergent. Then there exist two subsequences x_{n_k} and x_{m_k} which converge to different limits.

Sequence x_k is bounded: by Bolzano-Weierstrass there exists subsequence x_{n_k} with $x_{n_k} \to a$ as $k \to \infty$. The sequence x_k does not converge to a. Hence there exists $\epsilon > 0$ so that we have for infinitely many k that $|x_k - a| \ge \epsilon$. The sequence of these infinitely many x_k is bounded, hence it has a convergent subsequence x_{m_k} with $x_{m_k} \to b$ as $k \to \infty$, and we have $|b - a| \ge \epsilon$.

(b) $f\colon [a,b] \to \mathbb{R}$ is differentiable. f is neither increasing nor decreasing on $[a,b]$. Then there exists $x \in (a,b)$ with $f'(x) = 0$.

Method 1: Use Strict Monotonicity Theorem: For f differentiable on $[a,b]$ with $f'(x) \neq 0$ in (a,b) we have that either f is increasing on $[a,b]$, or that f is decreasing on $[a,b]$. Use contraposition.

Method 2: f is not increasing on $[a,b]$. Hence there exists $x, y \in [a,b]$ with $x < y$ and $f(x) \ge f(y)$. By the mean value theorem there exists $t \in (x,y)$ with $f'(t) = \frac{f(y)-f(x)}{y-x} \le 0$. In the same way: f is not decreasing on $[a,b]$ implies that there exists $s \in (a,b)$ with $f'(s) \le 0$. If $s = t$ then $f'(s) = 0$. If $s \neq t$ we have by the derivative intermediate value theorem that there exists p between s and t with $f'(p) = 0$.

3. (20 pts) Consider $f(x) = \ln x$ for $x > 0$.

(a) Find the Taylor polynomial $p_3(x) = \cdots + \cdots (x-a)^3$ about $a = 1$. Write down the remainder term $R_3(x) = f(x) - p_3(x)$ in **Lagrange form** and in **Cauchy form**.

$f'(x) = x^{-1}$, $f''(x) = -x^{-2}$, $f'''(x) = 2x^{-3}$, $f^{(4)}(x) = -6x^{-4}$, $f^{(k+1)}(x) = (-1)^k k! x^{-k-1}$

$p_3(x) = f(1) + f'(1)(x-1) + \cdots + f'''(1)(x-1)^3/3! = 0 + 1 \cdot (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

Lagrange remainder term: $R_3(x) = f^{(4)}(t)(x-1)^4/4! = -6t^{-4}(x-1)^4/24 = -\frac{1}{4}t^{-4}(x-1)^4$ with t between 1 and x

Cauchy remainder term: $R_3(x) = \int_1^x f^{(4)}(t)\frac{(x-t)^3}{3!}dt = \int_1^x -t^{-4}(x-t)^3 dt$

(b) Write down the Lagrange remainder term $R_n(x)$. Find an upper bound $\left|R_n(\frac{3}{4})\right| \le \cdots$. What happens with the upper bound for $n \to \infty$?

$R_n(x) = f^{(n+1)}(t)(x-1)^{n+1}/(n+1)! = (-1)^n n! t^{-n-1}(x-1)^{n+1}/(n+1)! = (-1)^n t^{-n-1}(x-1)^{n+1}/(n+1)$

upper bound: $\left|R_n(\frac{3}{4})\right| \le (\frac{3}{4})^{-n-1}(\frac{1}{4})^{n+1}/(n+1) = \frac{1}{n+1}(\frac{1}{3})^{n+1}$ which goes to zero as $n \to \infty$.

4. (10 pts) Assume $f\colon [a,b] \to \mathbb{R}$ satisfies for all $x, y \in [a,b]$ that $|f(x) - f(y)| \le L|x - y|$. Let $I := \int_a^b f(x)dx$

(a) Show that $I = f(p)(b-a)$ for some $p \in (a,b)$.
f is Lipschitz, hence f is continuous. **Method 1:** use the integral mean value theorem with $g(x) = 1$ to obtain the result. **Method 2:** Since f is continuous on $[a,b]$ it has a maximum at $\overline{x} \in [a,b]$ and a minimum at $\underline{x} \in [a,b]$. Then $(b-a)f(\underline{x}) \le I \le (b-a)f(\overline{x})$, and by the intermediate value theorem there exists $p \in (a,b)$ with $f(p) = \frac{I}{b-a}$.

(b) Let $Q := (b-a)f(\frac{a+b}{2})$ ("midpoint rule"). Show that $|Q - I| \le L(b-a)^2/2$.
Method 1: Using (a) we have $|Q-I| = \left|f(\frac{a+b}{x}) - f(p)\right|(b-a)$. Since $\frac{a+b}{2}$ is the midpoint of $[a,b]$ and $p \in (a,b)$ we have $\left|p - \frac{a+b}{2}\right| < \frac{b-a}{2}$. Using the Lipschitz property we get $|Q-I| \le L\frac{(b-a)}{2}(b-a)$.
Method 2: We have $I - Q = \int_a^b \left[f(x) - f(\frac{a+b}{2})\right] dx$, this gives the sharper result

$$|Q-I| \le \int_a^b \left|f(x) - f(\frac{a+b}{2})\right| dx \le \int_a^b L\left|x - \frac{a+b}{2}\right| dx = L\cdot 2\int_0^{(b-a)/2} t\,dt = L\cdot 2\cdot\left[\frac{t^2}{2}\right]_0^{(b-a)/2} = L\frac{(b-a)^2}{4}$$

5. (30 pts) The function $f\colon [0,4] \to \mathbb{R}$ satisfies for all $x, y \in [0,4]$ that $|f(x) - f(y)| \le 5\,|x-y|$.

(a) Show that f is uniformly continuous on $[0,4]$.
For a given $\epsilon > 0$ define $\delta := \epsilon/5$, then $|x-y| < \delta$ implies $|f(x) - f(y)| \le 5\,|x-y| < 5 \cdot \frac{\epsilon}{5} = \epsilon$.

(b) We are given $\epsilon > 0$. Find $\delta > 0$ (specify e.g. $\delta := \epsilon/17$) so that for a partition P with $|P| < \delta$ we have $U(f,P) - L(f,P) < \epsilon$.
Hint: Write $U(f,P) - L(f,P) = \sum_{j=1}^n (\cdots)(x_j - x_{j-1})$.
The partition P uses the points $0 = x_0 < x_1 < \cdots < x_n = 4$. The function f is continuous by (a). Hence it has on each subinterval $[x_{j-1}, x_j]$ a maximum at $\overline{m}_j$ and a minimum at $\underline{m}_j$, and $\left|\overline{m}_j - \underline{m}_j\right| \le x_j - x_{j-1} \le \delta$

$$U(f,P) - L(f,P) = \sum_{j=1}^n \left(f(\overline{m}_j) - f(\underline{m}_j)\right)(x_j - x_{j-1}) \le \sum_{j=1}^n 5\left|\overline{m}_j - \underline{m}_j\right|(x_j - x_{j-1}) \le 5\delta \underbrace{\sum_{j=1}^n (x_j - x_{j-1})}_{4} = 20\delta$$

Hence choosing $\delta := \epsilon/20$ implies $U(f,P) - L(f,P) < \epsilon$

(c) We want to find an approximation Q for $I := \int_0^4 f(x)dx$ with $|Q - I| \le 10^{-2}$. How can we do this?
Hint: Use a Riemann sum, specify the quadrature points you want to use. Then prove that $|Q - I| \le 10^{-2}$.
Use 5(b). For a partition P and any choice of points $s_j \in [x_{j-1}, x_j]$ we have $U(f,P) \ge R(f,P,S) \ge L(f,P)$ and also $U(f,P) \ge I \ge L(f,P)$. Therefore $U(f,P) - L(f,P) \le \epsilon$ implies $|I - R(f,P,S)| \le \epsilon$. Here $\epsilon = 10^{-2}$ is given. By (b): For a partition P with $|P| \le 10^{-2}/20 = \frac{1}{2000}$ we have $|I - R(f,P,S)| \le \epsilon = 10^{-2}$. Therefore we divide the interval $[0,4]$ into 2000 subintervals of equal size: $x_j = j/500$, $j = 0, \ldots, 2000$. We can use any $s_j \in [x_{j-1}, x_j]$, e.g., the right endpoint $s_j = x_j$. This gives the approximation $Q = \frac{1}{500}\sum_{j=1}^{500} f(j/500)$.
Alternatively, one can also use 4(b).