Defining the square root $b^{1/2}$ of a positive number b

For an ordered field we have (Proposition 1.10(b) in the notes)

$$x > 0 \text{ and } y < z \implies xy < xz \tag{1}$$

Proposition 1. *Let $y, z \in \mathbb{R}_+$. Then*
(i) *$y < z$ implies $y^2 < z^2$*
(ii) *$y \leq z$ implies $y^2 \leq z^2$*
(iii) *$y^2 > z^2$ implies $y > z$.*

Proof. For (i) we get with (1) $y \cdot y < y \cdot z < z \cdot z$. Including the equality case gives (ii). Statement (iii) is the same as (ii) (contraposition). □

Proposition 2. *Let $b \in \mathbb{R}_+$.*
(i) *For $p \in \mathbb{R}_+$ with $p^2 > b$ there exists $q < p$ with $q^2 > b$.*
(ii) *For $\tilde{p} \in \mathbb{R}_+$ with $\tilde{p}^2 < b$ there exists $\tilde{q} > \tilde{p}$ with $\tilde{q}^2 < b$.*

Proof. **(i):** Let $f(x) := x^2 - b$ and the tangent line $g(x) := f(p) + f'(p)(x - p) = p^2 - b + 2p(x - b)$. Define q such that $g(q) = 0$, i.e.,

$$q := p - \frac{p^2 - b}{2p}$$

Since $p^2 > b$ we have $q < p$. We need to show $q^2 > b$, i.e., $f(q) > 0$ [see e.g. page 7 in the notes for the proof of this].
(ii): Let $p := b/\tilde{p}$. Then $p^2 = b^2/\tilde{p}^2 > b^2/b = b$. From (i) we obtain $q < p$ with $q^2 > b$. Then $\tilde{q} := b/q > b/p = \tilde{p}$ and $\tilde{q}^2 = b^2/q^2 < b^2/b = b$. □

Proposition 3. *For $b \in \mathbb{R}_+$ there exists a unique $a \in \mathbb{R}_+$ with $a^2 = b$.*

Proof. Consider the set $A := \{x \in \mathbb{R}_+ : x^2 < b\}$.
The set A is nonempty: We have $s := b/(b+1) < 1$, hence $s^2 < s$. We also have $s < b$, hence $s^2 < s < b$, i.e., $s \in A$.
The set A is bounded from above: We have $t := b + 1 > 1$, hence $t^2 > t > b$. Let $x \in A$, then $x^2 < b < t^2$ which implies $x < t$ by Prop. 1(iii). Therefore t is an upper bound for the set A.
By the least upper bound property the set A has a least upper bound

$$a := \sup A \tag{2}$$

We now show that $a^2 = b$ using trichotomy:
Assume $a^2 < b$ holds. Then by Prop. 2(ii) there exists $\tilde{q} > a$ with $\tilde{q}^2 < b$, i.e., $\tilde{q} \in A$. Then a is not an upper bound of A, contradicting (2).
Assume $a^2 > b$ holds. Then by Prop. 2(i) there exists $q < a$ with $q^2 > b$. Let $x \in A$, then $x^2 < b < q^2$ which implies $x < q$ by Prop. 1(iii). Therefore q is an upper bound for the set A. But $q < a$ contradicts that a is the least upper bound.
Proof of uniqueness: This follows from Prop.1 (i). □

Note that $a^2 = 0$ implies $a = 0$. Hence:
For $b \in \mathbb{R}$ with $b \geq 0$ there exists a unique $a \in \mathbb{R}$, $a \geq 0$ such that $a^2 = b$. We use the notation $b^{1/2}$ for this number a.