Practice problems: Solutions

1. Approximate $y = (3.5)^{1/2}$ using the Taylor polynomial $p_2(x)$. Give an upper bound $|y - p_2(x)| \le \cdots$.
For $f(x) = x^{1/2}$ we use the Taylor polynomial about $x_0 = 4$: We have $f(x_0) = 2$, $f'(x_0) = \frac{1}{2}x_0^{-1/2} = \frac{1}{4}$, $f''(x_0) = -\frac{1}{4}x_0^{-3/2} = -\frac{1}{32}$, $f'''(x) = \frac{3}{8}x^{-5/2}$

$$\begin{aligned}
p_2(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 = 2 + \frac{1}{4}(x - x_0) - \frac{1}{64}(x - x_0)^2 \\
&= 2 - \frac{1}{8} - \frac{1}{64} \cdot \frac{1}{4} = 2 - \frac{33}{256} \\
f(x) - p(x) = R_3 &= \frac{1}{3!}\frac{3}{8}t^{-5/2}(x - x_0)^3 \\
|f(x) - p(x)| &\le \frac{1}{16}\left|t^{-5/2}\right|\left(\tfrac{1}{2}\right)^3 \quad \text{with } 3.5 < t < 4 \\
|f(x) - p(x)| &\le \frac{1}{16}(3.5)^{-5/2}\frac{1}{8}
\end{aligned}$$

2. We use the following Matlab command: `y = 1000.2 - 1000.1`
Give an upper bound for the relative error of the computed result
$\hat{y}_1 := fl(1000.2)$, $\hat{y}_2 := fl(1000.1)$, $\tilde{y} := \hat{y}_1 - \hat{y}_2$, $\hat{y} := fl(\tilde{y})$.
Let $\epsilon_{\hat{y}_1} = \frac{\hat{y}_1 - y_1}{y_1}$ etc. Then $|\epsilon_{\hat{y}_1}| \le \epsilon_M$, $|\epsilon_{\hat{y}_2}| \le \epsilon_M$, $|\epsilon_{\tilde{y}}| \le \frac{|y_1|}{|y_1 - y_2|}|\epsilon_{\hat{y}_1}| + \frac{|y_2|}{|y_1 - y_2|}|\epsilon_{\hat{y}_2}| \le 20003\epsilon_M$,
$|\epsilon_{\hat{y}}| \le |\epsilon_{\tilde{y}}| + \epsilon_M \le 20004\epsilon_M \approx 2 \cdot 10^{-12}$

3. We want to compute $y = e^{.001} - 1$ and use the Matlab code `y = exp(.001) - 1`

(a) Which operation (exp or subtraction) will cause a large magnification of the relative error? Find the magnification factor (condition number) for this operation, give the approximate answer as a number like $3 \cdot 10^7$. *Hint:* Use a Taylor approximation for $e^{.001}$ to evaluate your expression for the error.
Here $x := .001$, $y_1 := e^x$, $y := y_1 - 1$. The operation which causes the large error is the subtraction $y := 1 - y_1$ since $y_1 = e^{.001}$ is very close to 1. Recall that for $z := x + y$ we have $|\epsilon_{\tilde{z}}| \le \left|\frac{x}{x+y}\right||\epsilon_{\tilde{x}}| + \left|\frac{y}{x+y}\right||\epsilon_{\tilde{y}}|$. Applied to our case $y := y_1 - 1$ we get

$$|\epsilon_{\tilde{y}}| \le \left|\frac{y_1}{y_1 - 1}\right||\epsilon_{\tilde{y}}|, \qquad \frac{y_1}{y_1 - 1} = \left.\frac{e^x}{e^x - 1}\right|_{x=.001}.$$

(since there is no error present in the number 1). We now have to find the value of $\frac{e^x}{1-e^x}$ for $x = .001$. The Taylor series for e^x is $1 + x + \cdots$. Therefore we obtain using the leading term in the numerator and denominator

$$\frac{e^x}{e^x - 1} \approx \frac{1}{x} = \frac{1}{.001} = 1000$$

Therefore the magnification factor (condition number) is 10^3.

(b) Can we get a more accurate result if we evaluate the Taylor approximation $p_3(x)$ in Matlab?
Use the Taylor polynomial $p_n(x)$ about $x_0 = 0$ to approximate $f(x) = e^x - 1$: For $n = 3$ we get $f(x) \approx p_3(x) = 0 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$. We have for the absolute error with $t \in [0, .001]$

$$|f(x) - p_3(x)| = |R_4| = \left|e^t\frac{(x - x_0)^4}{4!}\right| \le e^{.001}\frac{(.001)^4}{4!} = e^{.001}\frac{10^{-12}}{24} \approx \frac{1}{2} \cdot 10^{-13} = 5 \cdot 10^{-14}$$

and for the relative error (using $|f(x)| \approx |x| = .001$)

$$\frac{|f(x) - p_3(x)|}{|f(x)|} \approx \frac{|f(x) - p_3(x)|}{.001} \le 10^3 \cdot 5 \cdot 10^{-14} = \boxed{5 \cdot 10^{-11}}$$

Hence using $p_3(x)$ causes an approximation error of about $5 \cdot 10^{-11}$. Therefore this will not give a smaller error than our original "naive code". We need to use more terms in the Taylor series, then we can obtain a more accurate result.

4. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 4 \\ 4 & 1 & 2 \end{bmatrix}$

(a) Apply Gaussian elimination using the pivot candidate with the largest absolute value to find the matrices L, U and the vector p .

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 1/2 & 2/7 & 1 \end{bmatrix}, U = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 7/4 & 7/2 \\ 0 & 0 & 2 \end{bmatrix}, p = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

(b) Use L, U, p to solve the linear system $Ax = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

Solving $Ly = \begin{bmatrix} b_{p_1} \\ b_{p_2} \\ b_{p_3} \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ gives $y = \begin{bmatrix} -1 \\ 9/4 \\ 6/7 \end{bmatrix}$, solving $Ux = y$ gives $x = \begin{bmatrix} -4/7 \\ 3/7 \\ 3/7 \end{bmatrix}$.

(c) We solve the linear system $Ax = \begin{bmatrix} 1 \\ 10 \\ 1000 \end{bmatrix}$ and find the solution vector x. Then we find out that we actually need the solution vector $\tilde{x}$ for the linear system $A\tilde{x} = \begin{bmatrix} -1 \\ 10 \\ 1000 \end{bmatrix}$. Find an upper bound $\frac{\|\tilde{x} - x\|_\infty}{\|x\|_\infty} \leq \cdots$ assuming $\left\|A^{-1}\right\|_\infty \leq 10$.

We have $Ax = b$ and $A\tilde{x} = \tilde{b}$ with $b = \begin{bmatrix} 1 \\ 10 \\ 1000 \end{bmatrix}$, $\tilde{b} = \begin{bmatrix} -1 \\ 10 \\ 1000 \end{bmatrix}$, hence $\frac{\|\tilde{b}-b\|_\infty}{\|b\|_\infty} = \frac{2}{1000}$. We have $\|A\|_\infty = 7$, hence

$$\frac{\|\tilde{x} - x\|_\infty}{\|x\|_\infty} \leq \|A\|_\infty \left\|A^{-1}\right\|_\infty \frac{\left\|\tilde{b} - b\right\|_\infty}{\|b\|_\infty} \leq 7 \cdot 10 \cdot \frac{2}{1000} = \frac{140}{1000} = .14$$