Solution of Assignment #1

Do all problems with pencil and paper unless I ask you to use Matlab.

1. Consider the vectors $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} -1 \\ t \\ 0 \end{bmatrix}$ where (i) $t = 1$, (ii) $t = -1$. For each case (i), (ii): Are the vectors linearly dependent? If yes, find c_1, c_2, c_3 which are not all zero such that $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{0}$. If no, prove that $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{0}$ implies $c_1 = c_2 = c_3 = 0$.
For (i) the vectors are linearly independent: $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{0}$ leads to the linear system

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the third equation we get $c_3 = 0$, and solving the first 2 equations then gives $c_1 = 0$, $c_2 = 0$.
For (ii) the vectors are linearly dependent: obviously $\vec{w} = -\vec{v}$, hence $c_1 = 0$, $c_2 = 1$, $c_3 = 1$ gives a linear combination which is $\vec{0}$.

2. Consider the subspace $U = \operatorname{span}\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} \}$. Select some of these vectors such that (i) the vectors are linearly independent and (ii) they span the same subspace U.
*Any two of the four vector*s will work, *except* vector 1 and vector 3 together.
E.g. let us choose the vectors $\vec{u}^{(1)} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\vec{u}^{(2)} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. **(i)** Show that the two vectors are linearly independent: For $c_1\vec{u}^{(1)} + c_2\vec{u}^{(2)} = \vec{0}$ the first two equations give $c_1 + c_2 = 0$ and $c_1 + 2c_2 = 0$ which implies $c_1 = 0$, $c_2 = 0$.
(ii) Show that $\operatorname{span}\{\vec{u}^{(1)}, \vec{u}^{(2)}\} = U$: We can write the remaining vectors as linear combinations of $\vec{u}^{(1)}, \vec{u}^{(2)}$: $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = (-1) \cdot \vec{u}^{(1)} + 0 \cdot \vec{u}^{(2)}$, $\begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} = 2 \cdot \vec{u}^{(1)} + 1 \cdot \vec{u}^{(2)}$.

3. Consider the matrix $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 4 & 1 & t & 1 \end{bmatrix}$ where (i) $t = 2$, (ii) $t = 1$. For each case (i), (ii):
Perform Gaussian elimination as far as possible, **choose the first nonzero element among the pivot candidates as pivot**. Is the matrix A singular or nonsingular?

- **If the matrix is singular:** use the resulting matrix from the elimination to find a vector $\vec{x} \neq \vec{0}$ such that $A\vec{x} = \vec{0}$.
- **If the matrix is nonsingular:** check that LU gives the expected result. Use L, U, p to solve the linear system $Ax = [2, 3, 2, 1]^\top$.
  Then solve the linear system in Matlab using `[L,U,p]=lu(A,'vector')` . Does Matlab get the same $L, U, \vec{p}$ as you did? Explain!

(i) After we are done with column 1:

$$L = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ \frac{3}{2} & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot \end{bmatrix}, \quad U = \begin{bmatrix} \textcircled{2} & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & \textcircled{$-\frac{1}{2}$} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -1 & 0 & -1 \end{bmatrix}, \quad \vec{p} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

After we are done with column 2:

$$L = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \frac{3}{2} & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \\ 2 & 2 & \cdot & \cdot \end{bmatrix}, \quad U = \begin{bmatrix} \textcircled{2} & 1 & 1 & 1 \\ 0 & \textcircled{$-\frac{1}{2}$} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \textcircled{1} & 0 \end{bmatrix}, \quad \vec{p} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix}$$

After we are done with column 3:

$$L = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \frac{3}{2} & \cdot & \cdot & \cdot \\ 2 & 2 & \cdot & \cdot \\ 0 & 0 & 0 & \cdot \end{bmatrix}, \quad U = \begin{bmatrix} \textcircled{2} & 1 & 1 & 1 \\ 0 & \textcircled{$-\frac{1}{2}$} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & \textcircled{1} \end{bmatrix}, \quad \vec{p} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$$

Since we obtain a matrix U with nonzero diagonal elements the matrix A is nonsingular. We obtain $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{2} & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, and LU gives $\begin{bmatrix} 2 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 4 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \text{row 2 of } A \\ \text{row 3 of } A \\ \text{row 4 of } A \\ \text{row 1 of } A \end{bmatrix}$. For solving the linear system we first solve $L\vec{y} = \begin{bmatrix} b_2 \\ b_3 \\ b_4 \\ b_1 \end{bmatrix}$ by forward substitution, yielding $\vec{y} = \begin{bmatrix} 3 \\ -\frac{5}{2} \\ 0 \\ 2 \end{bmatrix}$. Then we solve $U\vec{x} = \vec{y}$ using back substitution, yielding $\vec{x} = \begin{bmatrix} -1 \\ 3 \\ 0 \\ 2 \end{bmatrix}$.

In Matlab we use

```
A = [0 0 0 1; 2 1 1 1; 3 1 1 1; 4 1 2 1]; b = [2;3;2;1];
[L,U,p] = lu(A,'vector')
y = L\b(p)
x = U\y
```

This gives the same solution vector $\vec{x}$ as above. However, Matlab selects different pivots. It always selects the pivot candidate with the largest absolute value: for column 1 Matlab uses 4 as a pivot (instead of 2 as we did above). Therefore Matlab gets a different vector $\vec{p}$, different matrices L, U and a different vector $\vec{y}$.

(ii) After we are done with column 1:

$$L = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ \frac{3}{2} & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot \end{bmatrix}, \quad U = \begin{bmatrix} \textcircled{2} & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & \textcircled{$-\frac{1}{2}$} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -1 & -1 & -1 \end{bmatrix}, \quad \vec{p} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

After we are done with column 2:

$$L = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \frac{3}{2} & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \\ 2 & 2 & \cdot & \cdot \end{bmatrix}, \quad U = \begin{bmatrix} \textcircled{2} & 1 & 1 & 1 \\ 0 & \textcircled{-\frac{1}{2}} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \boxed{0} & 1 \\ 0 & 0 & \boxed{0} & 0 \end{bmatrix}, \quad \vec{p} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix}$$

We see that both pivot candidates (marked in red) are zero, hence the matrix is singular. Since the algorithm broke down in column 3 we can construct a vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ 1 \\ 0 \end{bmatrix}$ such that $A\vec{x} = \vec{0}$: For $U\vec{x} = \vec{0}$ (eq.3) and (eq.4) are obviously true. (eq.2) gives $x_2 = -1$, then (eq.1) gives $x_1 = 0$. Hence $\vec{x} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ satisfies $A\vec{x} = \vec{0}$.

4. Consider the three linear systems

$$\text{(i)} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \qquad \text{(ii)} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \qquad \text{(iii)} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

For each linear system do the following:

(a) Find the set of all $\vec{x} \in \mathbb{R}^2$ which satisfy the linear system.
(i) The unique solution $\vec{x} = \begin{bmatrix} \frac{5}{3} \\ \frac{2}{3} \end{bmatrix}$. **(ii)** There is no solution. **(iii)** There are infinitely many solutions: $\vec{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ with $x_2 \in \mathbb{R}$ arbitrary. Note that these points form a line through the point $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ with the direction vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

(b) Use Matlab to plot the two lines corresponding to the two equations together. See the instructions on the web page. Explain how you can see the solution set.

5. Consider the two linear systems

$$\text{(i)} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \qquad \text{(ii)} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

For each linear system do the following:

(a) Find the set of all $\vec{x} \in \mathbb{R}^3$ which satisfy the linear system by hand.
(i) Gaussian elimination yields the following linear system $U\vec{x} = \vec{y}$

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{5}{2} \\ 0 \end{bmatrix}$$

We can choose $x_3 \in \mathbb{R}$ arbitrarily, then (eq.2) yields $x_2 = -\frac{5}{3} + x_3$, then (eq.1) yields $x_1 = -\frac{1}{3} + x_3$. Hence we obtain $\vec{x} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{5}{3} \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ with $x_2 \in \mathbb{R}$arbitrary. Note that these points form a line through the point $\begin{bmatrix} -\frac{1}{3} \\ -\frac{5}{3} \\ 0 \end{bmatrix}$ with the direction vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(ii) Gaussian elimination yields the following linear system $U\vec{x} = \vec{y}$

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & \frac{7}{2} & -\frac{3}{2} \\ 0 & 0 & \frac{6}{7} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{5}{2} \\ \frac{10}{7} \end{bmatrix}$$

and back substitution gives the unique solution $\vec{x} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ \frac{5}{3} \end{bmatrix}$.

(b) Use Matlab to plot the three planes corresponding to the three equations together. See the instructions on the web page. Explain how you can see the solution set.