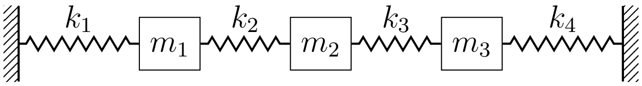
Solution Assignment #2, due Thursday, May 7

1. We consider three masses and four springs with spring constants k_1, k_2, k_3, k_4. The masses can only move horizontally. This is the picture at equilibrium:



Let $m_1 = m_2 = m_3 = 1$ and $k_1 = 2$, $k_2 = k_3 = 1$, $k_4 = 2$.

(a) We pull with horizontal forces F_1, F_2, F_3 at the three masses. Then the masses will move to new equilibrium positions. We want to know the resulting horizontal displacements x_1, x_2, x_3 of the masses from their original positions. Write down the linear system $A\vec{x} = \vec{F}$ with a 3×3 matrix A. Use Matlab to find the answer for $\vec{F} = [4, 2, 4]^\top$.

$$A = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}, \text{ solving } A\vec{x} = \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} \text{ gives } \vec{x} = \begin{bmatrix} \frac{5}{2} \\ \frac{7}{2} \\ \frac{5}{2} \end{bmatrix}$$

(b) Now we consider the time dependent problem with $\vec{F} = \vec{0}$. Find the eigenmodes of the form $\cos(\omega t)\vec{v}$ by hand (hint: the eigenvalues are small integers). Check your answer in Matlab using symbolic matrices.

$\det(A - \lambda I) = -\lambda^3 + 8\lambda^2 - 19\lambda + 12 = 0$ gives eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = 4$

the corresponding eigenvectors are $\vec{v}^{(1)} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}^{(2)} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}^{(3)} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

the eigenfrequencies are $\omega_j = \sqrt{\lambda_j}$, yielding the three eigenmodes

$$\boxed{\cos(t) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \qquad \cos(\sqrt{3}t) \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \qquad \cos(2t) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}$$

(c) Now consider the problem with $\vec{F} = \vec{0}$ and initial conditions

$$\vec{x}(0) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \qquad \vec{x}'(0) = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

Write down the general solution with parameters c_1, c_2, c_3 and d_1, d_2, d_3. Write down the linear system for $\vec{c}$ and for $\vec{d}$. Then use Matlab to find $\vec{c}, \vec{d}$.

The general solution is

$$\begin{aligned} \vec{x}(t) =& c_1 \cos(\omega_1 t)\vec{v}^{(1)} + c_2 \cos(\omega_2 t)\vec{v}^{(2)} + c_3 \cos(\omega_3 t)\vec{v}^{(3)} \\ &+ d_1 \sin(\omega_1 t)\vec{v}^{(1)} + d_2 \sin(\omega_2 t)\vec{v}^{(2)} + d_3 \sin(\omega_3 t)\vec{v}^{(3)} \\ \vec{x}'(t) =& -c_1\omega_1 \sin(\omega_1 t)\vec{v}^{(1)} - c_2\omega_2 \sin(\omega_2 t)\vec{v}^{(2)} - c_3\omega_3 \sin(\omega_3 t)\vec{v}^{(3)} \\ &+ \underbrace{d_1\omega_1}_{e_1} \cos(\omega_1 t)\vec{v}^{(1)} + \underbrace{d_2\omega_2}_{e_2} \cos(\omega_2 t)\vec{v}^{(2)} + \underbrace{d_3\omega_3}_{e_3} \cos(\omega_3 t)\vec{v}^{(3)} \end{aligned} \tag{1}$$

where we define $e_j := d_j\omega_j$. Plugging in $t = 0$ into these two equations gives with $V = \left[\vec{v}^{(1)}, \vec{v}^{(2)}, \vec{v}^{(3)}\right]$

$$\begin{aligned} \vec{x}(0) &= c_1\vec{v}^{(1)} + c_2\vec{v}^{(2)} + c_3\vec{v}^{(3)} = V\vec{c} \\ \vec{x}'(0) &= e_1\vec{v}^{(1)} + e_2\vec{v}^{(2)} + e_3\vec{v}^{(3)} = V\vec{e} \end{aligned}$$

so the initial conditions give the two linear systems $V\vec{c} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $V\vec{e} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$. Solving the linear systems gives $\vec{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{e} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ giving $\vec{d} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and the solution of the initial value problem

$$\boxed{\vec{x}(t) = \cos(2t) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \sin(t) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \sin(2t) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}$$

(d) Now consider the problem with $\vec{F} = [4, 2, 4]$ and the same initial conditions as (c). Find the solution of the initial value problem. You can use Matlab to solve the linear systems.

Now the general solution consists of the particular solution $\vec{x}_{\text{part}} = \begin{bmatrix} \frac{5}{2} \\ \frac{7}{2} \\ \frac{5}{2} \end{bmatrix}$ from (a) plus the general solution $\vec{x}_{\text{hom}}(t)$ given by (1):

$$\vec{x}(t) = \vec{x}_{\text{part}} + \vec{x}_{\text{hom}}(t)$$

Now the initial conditions give the two linear systems $\begin{bmatrix} \frac{5}{2} \\ \frac{7}{2} \\ \frac{5}{2} \end{bmatrix} + V\vec{c} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $V\vec{e} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$. Solving the linear systems gives $\vec{c} = \begin{bmatrix} -2 \\ 0 \\ \frac{1}{2} \end{bmatrix}$ and $\vec{e} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ giving $\vec{d} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and the solution of the initial value problem

$$\boxed{\vec{x}(t) = \begin{bmatrix} \frac{5}{2} \\ \frac{7}{2} \\ \frac{5}{2} \end{bmatrix} - 2\cos(t) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \frac{1}{2}\cos(2t) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \sin(t) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \sin(2t) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}$$

2. For the following matrices: Find a nonsingular matrix $V \in \mathbb{C}^{n\times n}$ and a matrix $B \in \mathbb{C}^{n\times n}$ in Jordan form (i.e., having Jordan boxes along diagonal) such that $AV = VB$. Do this by hand (hint: there are only two different eigenvalues, and one eigenvalue is easy to see). Note: Use $(A - \lambda I)\vec{w} = \vec{v}$ to find a generalized eigenvector $\vec{w}$; here the eigenvector $\vec{v}$ has to be carefully chosen so that a solution $\vec{w}$ exists.
In Matlab use `[V,D]=eig(A)` with symbolic matrices. Then use `[V,B]=jordan(A)` .

$$\text{(i) } A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 4 \\ 0 & -3 & -1 & -2 \\ 0 & 3 & -1 & 0 \end{bmatrix}, \qquad \text{(ii) } A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 4 \\ 1 & -3 & -1 & -2 \\ 0 & 3 & -1 & 0 \end{bmatrix}$$

For both matrices we obtain the same characteristic polynomial $p(\lambda) = (-2 - \lambda)\det \begin{bmatrix} 1-\lambda & 1 & 4 \\ -3 & -1-\lambda & 2 \\ 3 & -1 & -\lambda \end{bmatrix}$. Then $p(\lambda) = 0$ gives the eigenvalues $4, -2, -2, -2$.

Case (i): For $\lambda = 4$ we obtain

$$M = A - 4I = \begin{bmatrix} -6 & 0 & 0 & 0 \\ 0 & -3 & 1 & 4 \\ 0 & -3 & -5 & -2 \\ 0 & 3 & -1 & -4 \end{bmatrix} \xrightarrow{\text{row ech. form}} \begin{bmatrix} -6 & 0 & 0 & 0 \\ 0 & -3 & 1 & 4 \\ 0 & 0 & -6 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{basis for null } M} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

For $\lambda = -2$ we obtain

$$M = A + 2I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 4 \\ 0 & -3 & 1 & -2 \\ 0 & 3 & -1 & 2 \end{bmatrix} \xrightarrow{\text{row ech. form}} \begin{bmatrix} 0 & 3 & 1 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{basis for null } M} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

So we found 2 linearly independent eigenvectors for $\lambda = -2$. Hence we need to find one generalized eigenvector by solving $M\vec{w} = \vec{v}$ where $\vec{v}$ is chosen in $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}$ so that the linear system has a solution. Since the first row of M is all zeros this only works for $\vec{v} = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}$: Solving $M\vec{w} = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ gives as one possible solution $\vec{w} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$. Therefore we obtain 2 Jordan chains of length 1 (for $\lambda = 4$, $\lambda = -2$) and one Jordan chain of length 2 (for $\lambda = -2$):

$$V = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Case (ii): For $\lambda = 4$ we obtain

$$M = A - 4I = \begin{bmatrix} -6 & 0 & 0 & 0 \\ 1 & -3 & 1 & 4 \\ 1 & -3 & -5 & -2 \\ 0 & 3 & -1 & -4 \end{bmatrix} \xrightarrow{\text{row ech. form}} \begin{bmatrix} -6 & 0 & 0 & 0 \\ 0 & -3 & 1 & 4 \\ 0 & 0 & -6 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{basis for null } M} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \vec{v}^{(1)}$$

For $\lambda = -2$ we obtain

$$M = A + 2I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 4 \\ 1 & -3 & 1 & -2 \\ 0 & 3 & -1 & 2 \end{bmatrix} \xrightarrow{\text{row ech. form}} \begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & -6 & 0 & -6 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{basis for null } M} \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \vec{v}^{(2)}$$

So we found only 1 eigenvector $\vec{v}^{(2)}$ for $\lambda = -2$. Hence we need to find two generalized eigenvectors $\vec{v}^{(2,1)}, \vec{v}^{(2,2)}$ by solving $M\vec{v}^{(2,1)} = \vec{v}^{(2)}$ and $M\vec{v}^{(2,2)} = \vec{v}^{(2,1)}$: (note that these linear systems have infinitely many solutions, but we can just pick an arbitrary solution):

$$\text{For } M\vec{v}^{(2,1)} = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad \text{one solution is } \vec{v}^{(2,1)} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\text{For } M\vec{v}^{(2,2)} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad \text{one solution is } \vec{v}^{(2,2)} = \begin{bmatrix} -1 \\ \frac{1}{6} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

Therefore we obtain one Jordan chains of length 1 (for $\lambda = 4$) and one Jordan chain of length 3 (for $\lambda = -2$):

$$V = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & \frac{1}{6} \\ -1 & -1 & -1 & \frac{1}{2} \\ 1 & 1 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Note that there are infinitely many choices for $\vec{v}^{(2,1)}, \vec{v}^{(2,2)}$, so your solution or the one given by Matlab's `jordan` command may look different.