The determinant

Motivation: area of parallelograms, volume of parallepipeds

Two vectors in $\mathbb{R}^2$: "oriented area" of a parallelogram

Consider two vectors $a^{(1)}, a^{(2)} \in \mathbb{R}^2$ which are linearly independent. We say

- $a^{(1)}, a^{(2)}$ have **positive orientation** if $a^{(2)}$ is the left of $a^{(1)}$
- $a^{(1)}, a^{(2)}$ have **negative orientation** if $a^{(2)}$ is the right of $a^{(1)}$

The two vectors $a^{(1)}, a^{(2)}$ define the **parallelogram** consisting of the points $c_1 a^{(1)} + c_2 a^{(2)}$ with $c_1, c_2 \in [0,1]$.

We are interested in the **oriented area** D of the parallelogram:

$$D = \begin{cases} \text{area} & \text{if } a^{(1)}, a^{(2)} \text{have positive orientation} \\ -\text{area} & \text{if } a^{(1)}, a^{(2)} \text{have negative orientation} \end{cases}$$

We call this "oriented area" the **determinant of the matrix** $A = [a^{(1)}, a^{(2)}] \in \mathbb{R}^{2\times 2}$:

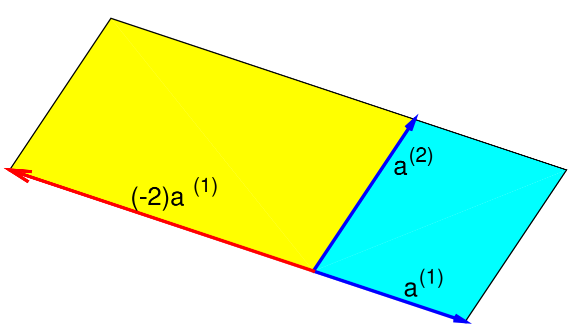
$$D = \det A$$

Note that we have the following properties:

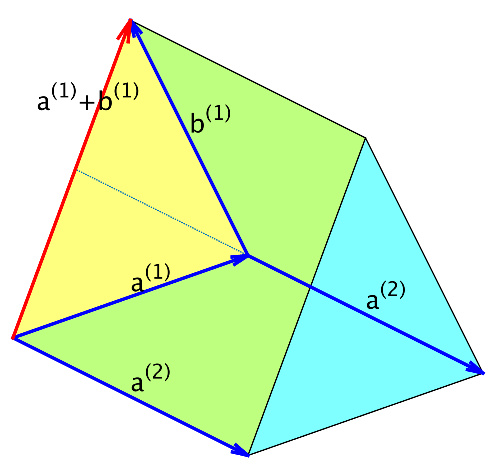
1. $\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$

2. $\det A = 0 \iff a^{(1)}, a^{(2)}$ are linearly dependent

3. $\det\left[ca^{(1)}, a^{(2)}\right] = c \det\left[a^{(1)}, a^{(2)}\right]$ for any scalar $c \in \mathbb{R}$
$\det\left[a^{(1)} + b^{(1)}, a^{(2)}\right] = \det[a^{(1)}, a^{(2)}] + \det[b^{(1)}, a^{(2)}]$
This means that the mapping $\det[a^{(1)}, a^{(2)}]$ is a **linear function of the first column** $a^{(1)}$
By the same argument, the mapping $\det[a^{(1)}, a^{(2)}]$ is a **linear function of the second column** $a^{(2)}$



$\det\left[(-2)a^{(1)}, a^{(2)}\right] = (-2)\cdot\det\left[a^{(1)}, a^{(2)}\right]$
oriented area of yellow parallelogram
= (-2)(oriented area of cyan parallelogram)



$\det\left[a^{(1)} + b^{(1)}, a^{(2)}\right] = \det\left[a^{(1)}, a^{(2)}\right] + \det\left[b^{(1)}, a^{(2)}\right]$
area of yellow parallelogram
= sum of areas of two cyan parallelograms

Three vectors in $\mathbb{R}^3$: "oriented volume" of a parallelepiped

Consider three vectors $a^{(1)}, a^{(2)}, a^{(3)} \in \mathbb{R}^3$ which are linearly independent. We say (somewhat imprecisely)

- $a^{(1)}, a^{(2)}, a^{(3)}$ have **positive orientation** if they are arranged according **three finger rule** (like thumb, index finger, middle finger of the right hand)
- $a^{(1)}, a^{(2)}, a^{(3)}$ have **negative orientation** if they are arranged in the opposite way (i.e., $a^{(1)}, a^{(2)}, -a^{(3)}$ have positive orientation)

Three vectors $a^{(1)}, a^{(2)}, a^{(3)} \in \mathbb{R}^3$ form a **parallelepiped** consisting of the points $c_1 a^{(1)} + c_2 a^{(2)} + c_3 a^{(3)}$ with $c_1, c_2, c_3 \in [0,1]$.

We are interested in the **oriented volume** D of the parallelepiped:

$$D = \begin{cases} \text{volume} & \text{if } a^{(1)}, a^{(2)}, a^{(3)} \text{have positive orientation} \\ -\text{volume} & \text{if } a^{(1)}, a^{(2)}, a^{(3)} \text{have negative orientation} \end{cases}$$

We call this "oriented volume" the **determinant of the matrix** $A = [a^{(1)}, a^{(2)}, a^{(3)}] \in \mathbb{R}$:

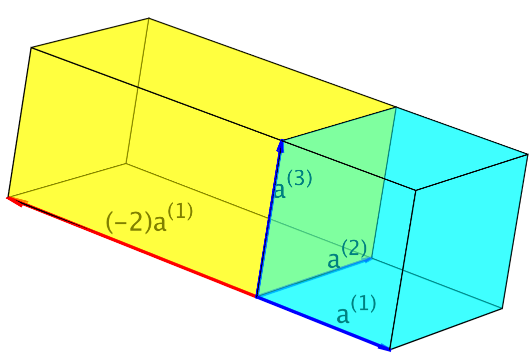
$$D = \det(A)$$

Note that we have the following properties:

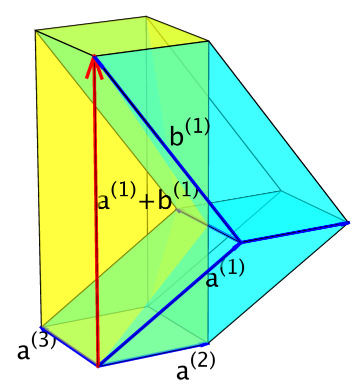
1. $\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$

2. $\det A = 0 \iff a^{(1)}, a^{(2)}, a^{(3)}$ are linearly dependent

3. $\det\left[c a^{(1)}, a^{(2)}, a^{(3)}\right] = c \det\left[a^{(1)}, a^{(2)}, a^{(3)}\right]$ for any scalar $c \in \mathbb{R}$
$\det\left[a^{(1)} + b^{(1)}, a^{(2)}, a^{(3)}\right] = \det[a^{(1)}, a^{(2)}, a^{(3)}] + \det[b^{(1)}, a^{(2)}, a^{(3)}]$
This means that $\det[a^{(1)}, a^{(2)}, a^{(3)}]$ is a **linear function of the first column** $a^{(1)}$
$\det[a^{(1)}, a^{(2)}, a^{(3)}]$ is a **linear function of the second column** $a^{(2)}$
$\det[a^{(1)}, a^{(2)}, a^{(3)}]$ is a **linear function of the third column** $a^{(3)}$



$\det\left[(-2)a^{(1)}, a^{(2)}, a^{(3)}\right] = (-2) \cdot \det\left[a^{(1)}, a^{(2)}, a^{(3)}\right]$
oriented volume of yellow parallelepiped
= (-2)(oriented volume of cyan parallelepiped)



$\det\left[a^{(1)} + b^{(1)}, a^{(2)}, a^{(3)}\right] = \det\left[a^{(1)}, a^{(2)}, a^{(3)}\right] + \det\left[b^{(1)}, a^{(2)}, a^{(3)}\right]$
volume of yellow parallelepiped
= sum of volumes of two cyan parallelepipeds

Abstract definition

Consider the matrix $A = [a^{(1)}, \dots, a^{(n)}] \in \mathbb{R}^{n\times n}$. We want to find a function $\det A$ with the following three properties

1. $\det I = 1$ where I is the $n \times n$ idenitity matrix
2. $\det A = 0 \iff a^{(1)}, \dots, a^{(n)}$ are linearly dependent
3. $\det\left[a^{(1)}, \dots, a^{(n)}\right]$ is a linear function of $a^{(1)}$

   $\vdots$

   is a linear function of $a^{(n)}$

Lemma 1. *Assume the function* $\det\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ *satisfies (1.), (2.), (3.), then we have for* $A \in \mathbb{R}^{n\times n}$ *and* $c \in \mathbb{R}$

$$\det[a^{(1)}, a^{(2)}, \dots, a^{(n)}] = \det\left[a^{(1)}, a^{(2)} + ca^{(1)}, a^{(3)}, \dots, a^{(n)}\right] \tag{1}$$

$$\det\left[a^{(1)}, a^{(2)}, \dots, a^{(n)}\right] = -\det[a^{(2)}, a^{(1)}, a^{(3)}, \dots, a^{(n)}] \tag{2}$$

Proof. Note that $\det[a^{(1)}, ca^{(1)}, a^{(3)}, \dots, a^{(n)}] = 0$ by property (3.). Hence (1) follows from property (2.).
In order to prove (2) we use (writing just "$\dots$" for "$a^{(3)}, \dots, a^{(n)}$") property (1) □

This means

- adding a multiple of one column to another column does not change the determinant
- swapping two columns changes the sign of the determinant

It is still not clear whether we can we find such a function, or whether there are multiple such functions.

Theorem 1. *There is a unique function* $\det\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ *which satisfies properties (1.), (2.), (3.).*

Induction Proof of Theorem 1

Case $n = 1$

For a 1×1 matrix $A = [a_{11}]$ we have

$$\det[a_{11}] \overset{(3.)}{=} a_{11}\det[1] \overset{(1.)}{=} a_{11}$$

and this function $\det[a_{11}] = a_{11}$ satisfies properties (1.), (2.), (3.).

Induction step: assuming result for $n-1$ show result for n

We assume that there is a unique function $\det\colon A^{(n-1)\times(n-1)} \to \mathbb{R}$ satisfying properties (1.), (2.), (3.). We want to show that there is a unique function $\det\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ satisfying properties (1.), (2.), (3.).

Claim 1. For any function $\det\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ satisfying (1.), (2.), (3.) there holds

$$\det \underbrace{\begin{bmatrix} 0 & a_{11} & \cdots & a_{1,n-1} \\ \vdots & \vdots & & \vdots \\ 0 & a_{j-1,1} & \cdots & a_{j-1,n-1} \\ 1 & 0 & \cdots & 0 \\ 0 & a_{j,1} & \cdots & a_{j,n-1} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n-1,1} & \cdots & a_{n-1,n-1} \end{bmatrix}}_{\tilde{A}} = (-1)^{j-1}\det \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1,n-1} \\ \vdots & & \vdots \\ a_{1,n-1} & \cdots & a_{n-1,n-1} \end{bmatrix}}_{A}$$

Proof. The function $\tilde{\det}\colon \mathbb{R}^{(n-1)\times(n-1} \to \mathbb{R}$, $\begin{bmatrix} a_{11} & \cdots & a_{1,n-1} \\ \vdots & & \vdots \\ a_{1,n-1} & \cdots & a_{n-1,n-1} \end{bmatrix} \mapsto (-1)^{j-1}\det\tilde{A}$ satisfies properties (1.), (2.),(3.):
Property (1.): $\tilde{\det}(I) = (-1)^{j-1}\det[e^{(j)}, e^{(1)}, \ldots, e^{(j-1)}, e^{(j+1)}, \ldots, e^{(n)}]$ where $e^{(k)}$ denotes the kth column of the $n\times n$ identity matrix. We can change the matrix $[e^{(j)}, e^{(1)}, \ldots, e^{(j-1)}, e^{(j+1)}, \ldots, e^{(n)}]$ with $(j-1)$ column interchanges to the $n\times n$ identity matrix. Hence we get from (2) and (1.) that

$$\det[e^{(j)}, e^{(1)}, \ldots, e^{(j-1)}, e^{(j+1)}, \ldots, e^{(n)}] = (-1)^{j-1}\cdot 1$$

Property (2.): The columns of the matrix $\tilde{A}$ are linearly independent $\iff$ the columns of the matrix A are linearly independent.
Property (3.): We obtain $\tilde{\det}[a^{(1)}+b^{(1)}, a^{(2)}, \ldots, a^{(n-1)}] = \tilde{\det}[a^{(1)}, a^{(2)}, \ldots, a^{(n-1)}] + \tilde{\det}[b^{(1)}, a^{(2)}, \ldots, a^{(n-1)}]$ since we have property (3.) for det on $\mathbb{R}^{n\times n}$.
Since by the induction there is a unique function $\det\colon \mathbb{R}^{n\times n}\to\mathbb{R}$ satisfying (1.), (2.), (3.) we must have $\tilde{\det} = \det$. ☐

Notation: For $A\in\mathbb{R}^{n\times n}$ **let** $A_{[ij]}\in\mathbb{R}^{(n-1)\times(n-1)}$ **denote the matrix with row** i **and column** j **removed**.

Claim 2. For any function $\det\colon \mathbb{R}^{n\times n}\to\mathbb{R}$ satisfying (1.), (2.), (3.) there holds

$$\boxed{\det A = a_{11}\det A_{[11]} - a_{21}\det A_{[21]} + \cdots + (-1)^{n-1}a_{n1}\det A_{[n,1]}} \tag{3}$$

Proof. We can write the first column $a^{(1)} = a_{11}e^{(1)} + \cdots + a_{n1}e^{(n)}$. By property (3.) the function det is linear in the first column, hence

$$\det A = a_{11}\det\left[e^{(1)}, a^{(2)}, \ldots, a^{(n)}\right] + \cdots + a_{n1}\det\left[e^{(n)}, a^{(2)}, \ldots, a^{(n)}\right]$$

Consider the matrix $[e^{(1)}, a^{(2)}, \ldots, a^{(n)}]$: If we subtract a_{12} times column 1 from column 2, ..., subtract a_{1n} times column 1 from column n we obtain the matrix $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ which must have the same determinant by (1). Therefore claim 1 givesdet $\left[e^{(1)}, a^{(2)}, \ldots, a^{(n)}\right] = \det A_{[11]}$. We obtain in the same way

$$\det\left[e^{(j)}, a^{(2)}, \ldots, a^{(n)}\right] = (-1)^{j-1}\det A_{[j1]}$$

☐

Note that we have obtained a formula for computing the determinant of an $n\times n$ matrix using determinants of $(n-1)\times(n-1)$ matrices. Therefore we have obtained a unique definition for $\det A$:

- for $n=1$: $\det[a_{11}] = a_{11}$
- for $n=2$: $\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}\det[a_{22}] - a_{21}\det[a_{12}] = a_{11}a_{22} - a_{21}a_{12}$
- for $n=3$: $\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\underbrace{\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}}_{(a_{22}a_{33}-a_{23}a_{32})} - a_{21}\underbrace{\det\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}}_{(a_{12}a_{33}-a_{12}a_{32})} + a_{31}\underbrace{\det\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}}_{(a_{12}a_{23}-a_{13}a_{22})}$

Example:

$$\det\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 1\cdot\underbrace{\det\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}}_{45-48} - 4\cdot\underbrace{\det\begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}}_{18-24} + 7\cdot\underbrace{\det\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}}_{12-15} = (-3)+24-21 = 0$$

Note that for $n = 3$ we obtain a sum of $3! = 6$ terms. For $n = 4$ we obtain $4! = 24$ terms. For a matrix $A \in \mathbb{R}^{n\times n}$ **applying the recursion formula gives a sum of $n!$ terms**. Unfortunately $n!$ grows very rapidly with increasing n which makes it impractical to use this method for $n > 3$.

Fortunately there is a more efficient way to compute the determinant: We can use column operations and column interchanges to reduce a matrix A to triangular form (in the same way we used row operations in Gaussian elimination): (here we show the pivot candidates in red)

$$A = \begin{bmatrix} * & * & * & * & * \\ * & \cdots & \cdots & \cdots & * \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ * & \cdots & \cdots & \cdots & * \end{bmatrix} \to \begin{bmatrix} \circledast & 0 & \cdots & \cdots & 0 \\ * & * & \cdots & \cdots & * \\ \vdots & * & \cdots & \cdots & * \\ \vdots & \vdots & & & \vdots \\ * & * & \cdots & \cdots & * \end{bmatrix} \to \begin{bmatrix} \circledast & 0 & \cdots & \cdots & 0 \\ * & \circledast & 0 & \cdots & 0 \\ * & * & * & \cdots & * \\ \vdots & \vdots & & & \vdots \\ * & * & \cdots & \cdots & * \end{bmatrix} \to \cdots \to \begin{bmatrix} \circledast & 0 & \cdots & \cdots & 0 \\ * & \circledast & 0 & \cdots & 0 \\ * & * & \circledast & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ * & * & \cdots & * & \circledast \end{bmatrix} = U$$

There are two cases:

- We end up with a triangular matrix U which has nonzero elements on the diagonal. Each column interchange switches the sign of the determinant. The column operations do not change the sign of the determinant. By (3) we obtain $\det U = u_{11}u_{22}\cdots u_{nn}$. If we used k column interchanges we obtain

$$\boxed{\det A = (-1)^k u_{11}u_{22}\cdots u_{nn}} \tag{4}$$

- The algorithm breaks down in row $j \in \{1,\ldots,n\}$ since all pivot candidates are zero. In this case the matrix is singular and $\boxed{\det A = 0}$.

Note:

- Since $\det A = \det A^\top$ (see below) we can just as well use the standard Gaussian elimination with row operations (instead of column operations), and (4) still holds (where U is the resulting upper triangular matrix, and k is the number of row interchanges during the elimination).
- Since Gaussian elimination takes $\frac{1}{3}n^3 + O(n^2)$ operations this method is much faster than the recursion formula which takes $n!$ operations.

To finish the proof of Theorem 1 we must prove that our function det given by 3 really satisfies properties (1.), (2.), (3.) for $A \in \mathbb{R}^{n\times n}$. [I skip this proof here.]

Additional properties of the determinant

Instead of column 1 **we can use any column j for the recursion formula:** (this follows since each swap of columns switches the sign of the determinant)

$$\boxed{\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{[ij]}} \tag{5}$$

Instead of columns **we can also use rows for the recursion formula:** Using row i we obtain

$$\boxed{\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{[ij]}} \tag{6}$$

Sketch of proof: For $A \in \mathbb{R}^{n\times n}$ define $\tilde{\det} A := \sum_{j=1}^{n} (-1)^{i+j} a_{1j} \det A_{[ij]}$. We then show that $\tilde{\det} A$ satisfies properties (1.), (2.), (3.). I skip the details here.

This formula shows that we have the following property:

- detA is a linear function of each row of the matrix

Therefore it follows that $\tilde{\det} A := \det A^\top$ also satisfies properties (1.), (2.), (3.). Hence we must have

Theorem 2. $\det A = \det A^\top$

The following result can be shown in a similar way:

Theorem 3. *For $A, B \in \mathbb{R}^{n\times n}$ we have* $\det(AB) = (\det A)\cdot(\det B)$.

Proof. For a fixed matrix $B \in \mathbb{R}^{n\times n}$ let us define $\tilde{\det} A := \det(AB)/\det(B)$. We can then show that the function $\tilde{\det}\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ satisfies properties (1.), (2.), (3.). I skip the details. $\square$

Theorem 4. *Assume that $A \in \mathbb{R}^{n\times n}$ is nonsingular.*

- *Let $b \in \mathbb{R}^n$. Then the linear system $Ax = b$ has the solution $x \in \mathbb{R}^n$ given by **"CRAMER's rule"**:*

$$x_i = \frac{\det\left[a^{(1)},\dots,a^{(i-1)},b,a^{(i+1)},\dots,a^{(n)}\right]}{\det A}$$

- *$M = A^{-1}$ has the elements*
-

$$M_{ij} = \frac{(-1)^{i+j}\det A_{[ji]}}{\det A}$$

Note: *This uses $A_{[ji]}$ and not $A_{[ij]}$!!*

Proof. Since $Ax = b$ the columns $A = [a^{(1)},\dots,a^{(n)}]$ satisfy

$$x_1a^{(1)}+\cdots+x_na^{(n)} = b$$
$$x_1a^{(1)}+\cdots+x_{i-1}a^{(i-1)}+1\cdot\left(x_ia^{(i)}-b\right)+x_{i+1}a^{(i+1)}+\cdots+x_na^{(n)} = \vec{0}$$

hence the matrix $\left[a^{(1)},\dots,a^{(i-1)},x_ia^{(i)}-b,a^{(i+1)},\dots,a^{(n)}\right]$ has linearly dependent columns and therefore determinant zero. By property (3.) for column i we therefore have

$$0 = x_i\det\left[a^{(1)},\dots,a^{(n)}\right]-\det\left[a^{(1)},\dots,a^{(i-1)},b,a^{(i+1)},\dots,a^{(n)}\right].$$

The columns of $M = [m^{(1)},\dots,m^{(n)}]$ satisfy $Am^{(j)} = e^{(j)}$ where $e^{(j)}$ is the jth column of the identity matrix. Hence by Cramer's rule

$$\left(m^{(j)}\right)_i = m_{ij} = \frac{\det\left[a^{(1)},\dots,a^{(i-1)},e^{(j)},a^{(i+1)},\dots,a^{(n)}\right]}{\det A} = \frac{(-1)^{i+j}\det A_{[ji]}}{\det A}$$

by using (5) with column i to find the determinant. $\square$

Note that these formulas are completely unpractical for $n > 3$ since $\det A$ consists $n!$ terms. If we use Gaussian elimination we can find x and A^{-1} using $cn^3 + O(n^2)$ operations.