Solution for practice problems

1. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 2 & 2 & 1 \\ 3 & 4 & 2 \end{bmatrix}$

(a) Perform Gaussian elimination to find the row echelon form U, the matrix L of multipliers, and the permutation vector p. **Always use the first available pivot candidate.**
We obtain

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad p = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix}$$

Note that $\operatorname{rank} A = 2$, and the numbers of the basic variables are $q_1 = 1$, $q_2 = 2$.

(b) Find a basis for range A: Use columns q_1, q_2 of A: $\begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 2 \\ 4 \end{bmatrix}$

(c) Find a basis for null A: Solve $U \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, this gives $\begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$

(d) Find a basis for range $A^\top$: **Answer 1:** Use rows 1,2 of U: $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}$

Answer 2: Use rows p_1, p_2 of A: $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

(e) Find a basis for null $A^\top$:

First vector: Solve $L^\top u = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, this gives $u = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, then let $\begin{matrix} w_{p_1} = u_1 \\ \vdots \\ w_{p_4} = u_4 \end{matrix}$, this gives

$$w = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Second vector: Solve $L^\top u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, this gives $u = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, then let $\begin{matrix} w_{p_1} = u_1 \\ \vdots \\ w_{p_4} = u_4 \end{matrix}$, this gives

$$w = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Hence the basis is given by the vectors $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

(f) Consider a linear system $Ax = b$ where $b \in \mathbb{R}^4$ is given. How many conditions does the vector b have to satisfy so that a solution exists? State these conditions (using your earlier results).
The vector b has to be orthogonal on all vectors in $\operatorname{null} A^\top$. Using the basis from (e) we obtain **two conditions:**

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot b = 0, \qquad \text{i.e.,} \quad -2b_1 + b_2 = 0$$

$$\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \cdot b = 0, \qquad \text{i.e.,} \quad -b_1 - b_3 + b_4 = 0$$

(g) Consider the linear system $Ax = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$. Find the general solution.

Find a particular solution: Find new right hand side vector y by solving $Ly = \begin{bmatrix} b_{p_1} \\ \vdots \\ b_{p_4} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, this gives $y = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$. Note that y_3 and y_4 are 0, hence there exists a solution. Then set the free variables to zero and solve $U \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = y$, this gives the particular solution $x_p = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

The general solution is given by x_p plus an arbitrary vector from $\operatorname{null} A$. Using (c) we obtain the **general solution**

$$x = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}, \qquad c \in \mathbb{R} \text{ arbitrary}$$

2. Assume we have a matrix $A \in \mathbb{R}^{2\times 4}$

(a) What are the possible values for $r = \operatorname{rank} A$? Give an example matrix A for each case!
For $A \in \mathbb{R}^{m\times n}$ the rank r is between 0 and $\min\{m, n\}$.
Here the matrix has two rows. The rank is the dimension of the row space. Therefore we can have $r = 0$, $r = 1$, $r = 2$.

Example for $r = 0$: zero linearly independent row vectors: $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Example for $r = 1$: one linearly independent row vectors: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix}$

Example for $r = 2$: two linearly independent row vectors: $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

(b) Assume rank $A = 1$. What is the dimension of the spaces range A, null A, range $A^\top$, $A^\top$?
For $A \in \mathbb{R}^{m\times n}$ we have

$$\begin{aligned} \dim \operatorname{range} A &= r, & \dim \operatorname{null} A^\top &= n - r \\ \dim \operatorname{range} A^\top &= r, & \dim \operatorname{null} A &= m - r \end{aligned}$$

(Recall that range A, null $A^\top$ are orthogonal complements in $\mathbb{R}^n$, and range $A^\top$, null A are orthogonal complements in $\mathbb{R}^m$.)
Here we obtain

$$\begin{aligned} \dim \operatorname{range} A &= 1, & \dim \operatorname{null} A^\top &= 1 \\ \dim \operatorname{range} A^\top &= 1, & \dim \operatorname{null} A &= 3 \end{aligned}$$

3. For a matrix $A \in \mathbb{R}^{3\times 5}$ we obtain the row echelon form $U = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and the permuation vector $p = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$.
DO NOT find the matrix A!

(a) Which of the columns of the matrix A form a basis for the column space?
The basic variables are x_1, x_3. The free variables are x_2, x_4, x_5. Hence **column 1 and column 3** form a basis for the column space.

(b) Which of the rows of the matrix A form a basis for the row space?
Rows $p_1, \dots, p_r$ of A form a basis for the row space: **Row 2 and row 3** of the matrix A form a basis for the row space.

(c) Find a basis for the orthogonal complement of the row space.
The row space corresponds to range $A^\top$. The orthogonal complement is A. Since the free variables are x_2, x_4, x_5 we have to solve three linear systems:
$U[x_1, 1, x_3, 0, 0]^\top = [0, 0, 0]^\top$ gives $x = [-\frac{1}{2}, 1, 0, 0, 0]^\top$
$U[x_1, 0, x_3, 1, 0]^\top = [0, 0, 0]^\top$ gives $x = [-1, 0, 1, 1, 0]^\top$
$U[x_1, 0, x_3, 0, 1]^\top = [0, 0, 0]^\top$ gives $x = [-1, 0, 2, 0, 1]^\top$
Note that $r = \operatorname{rank} A = 2$. Hence the row space range $A^\top$ is a 2-dimensional subspace of $\mathbb{R}^5$. The orthogonal complement has dimension 3, hence we need 3 basis vectors.
We obtain that the vectors $[-\frac{1}{2}, 1, 0, 0, 0]^\top$, $[-1, 0, 1, 1, 0]^\top$, $[-1, 0, 2, 0, 1]^\top$ form a basis for the orthogonal complement.