Solution of practice problems

1. We are given the data values $\begin{array}{c|cccc} t_j & -2 & -1 & 1 & 2 \\ \hline y_j & 3 & 1 & 2 & 4 \end{array}$. Find the least squares fit with a function of the form $y = c_1 t + c_2 |t|$.

We have $A = \begin{bmatrix} t_1 & |t_1| \\ \vdots & \vdots \\ t_4 & |t_4| \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}$. We use the normal equations $(A^\top A)c = (A^\top y)$ which gives the linear system $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \end{bmatrix}$. Hence $c = \begin{bmatrix} 0.3 \\ 1.7 \end{bmatrix}$, i.e., the least squares fit function is $y = 0.3t + 1.7\,|t|$.

2.

(a) For the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ find a decomposition $A = P \begin{bmatrix} 1 & s_{12} & s_{13} \\ 0 & 1 & s_{23} \\ 0 & 0 & 1 \end{bmatrix}$ where the columns of the matrix $P \in \mathbb{R}^{4\times 3}$ are orthogonal on each other.

Gram-Schmidt process: $p^{(1)} = a^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $p^{(2)} = a^{(2)} - \underbrace{\frac{p^{(1)} \cdot a^{(2)}}{p^{(1)} \cdot p^{(1)}}}_{s_{12}} p^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$,

$$p^{(3)} = a^{(3)} - \underbrace{\frac{p^{(1)} \cdot a^{(3)}}{p^{(1)} \cdot p^{(1)}}}_{s_{13}} p^{(1)} - \underbrace{\frac{p^{(2)} \cdot a^{(3)}}{p^{(2)} \cdot p^{(2)}}}_{s_{23}} p^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{0}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{3/2} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

Hence we obtain $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{3} \\ 1 & \frac{1}{2} & -\frac{1}{3} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$

(b) For a *different* matrix A we obtain the decomposition $A = \underbrace{\begin{bmatrix} -1 & 2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}}_{S}$ where the columns of the matrix P are orthogonal on each other. Use this to find $c \in \mathbb{R}^2$ such that $\left\| Ac - \begin{bmatrix} 5 \\ 2 \\ 5 \end{bmatrix} \right\|$ is minimal. DO **NOT** TRY TO FIND THE MATRIX A!

Here $p^{(1)} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$, $p^{(2)} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. Find $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$: $d_1 = \frac{p^{(1)} \cdot b}{p^{(1)} \cdot p^{(1)}} = \frac{9}{9} = 1$, $d_2 = \frac{p^{(2)} \cdot b}{p^{(2)} \cdot p^{(2)}} = \frac{18}{9} = 2$.

Then solve $Sc = d$: $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ which gives $c = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

3. Use the expansion formula to find $\det \begin{bmatrix} 0 & 2 & 3 & 5 \\ 2 & 7 & 8 & 9 \\ 0 & 2 & 3 & 1 \\ 0 & 4 & 0 & 0 \end{bmatrix}$. Hint: try to pick convenient rows or columns.

Method 1: First use column 1: $\det A = -2 \cdot \det \begin{bmatrix} 2 & 3 & 5 \\ 2 & 3 & 1 \\ 4 & 0 & 0 \end{bmatrix}$. Then use row 3: $\det \begin{bmatrix} 2 & 3 & 5 \\ 2 & 3 & 1 \\ 4 & 0 & 0 \end{bmatrix} =$ $4 \cdot \det \begin{bmatrix} 3 & 5 \\ 3 & 1 \end{bmatrix}$. Hence $\det A = (-2) \cdot 4 \cdot (3 - 15) = 96$.

Method 2: First use row 4: $\det A = 4 \cdot \det \begin{bmatrix} 0 & 3 & 5 \\ 2 & 8 & 9 \\ 0 & 3 & 1 \end{bmatrix}$. Then use column 1: $\det \begin{bmatrix} 0 & 3 & 5 \\ 2 & 8 & 9 \\ 0 & 3 & 1 \end{bmatrix} =$ $(-2) \cdot \det \begin{bmatrix} 3 & 5 \\ 3 & 1 \end{bmatrix}$. Hence $\det A = (-2) \cdot 4 \cdot (3 - 15) = 96$.

4. Let $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & -8 & 5 & 5 \\ 1 & -5 & 2 & 5 \\ 1 & -5 & 5 & 2 \end{bmatrix}$. DO **NOT** TRY TO FIND THE CHARACTERISTIC POLYNOMIAL!

(a) Write down the matrix $M = A - \lambda I$. Look at this matrix and try to guess a value λ which makes M singular (without using $\det M$). Find a basis for the eigenspace for this eigenvalue.

$M = A - \lambda I = \begin{bmatrix} 2-\lambda & 0 & 0 & 0 \\ 1 & -8-\lambda & 5 & 5 \\ 1 & -5 & 2-\lambda & 5 \\ 1 & -5 & 5 & 2-\lambda \end{bmatrix}$. We see that for $\lambda = 2$ the first row is all zeros and M must be singular. Now we have to find a basis for the null space of $M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -10 & 5 & 5 \\ 1 & -5 & 0 & 5 \\ \boxed{1} & -5 & 5 & 0 \end{bmatrix}$: find the row echelon form:

$$\begin{bmatrix} \boxed{1} & -5 & 5 & 0 \\ 1 & -10 & 5 & 5 \\ 1 & \boxed{-5} & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & -5 & 5 & 0 \\ 0 & \boxed{-5} & 0 & 5 \\ 0 & 0 & \boxed{-5} & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U.$$ Hence $\operatorname{rank} M = 3$ and $\dim \operatorname{null} A = 1$.

Solving $Uv = \vec{0}$ with $v_4 = 1$ gives $v = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

(b) Find a basis for the eigenspace for $\lambda = -3$. We now have $M = A + 3I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -5 & 5 & 5 \\ 1 & -5 & 5 & 5 \\ 1 & -5 & 5 & 5 \end{bmatrix}$ and we have to find a basis for the null space. We find the row echelon form:

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{-5} & 5 & 5 \\ 0 & -5 & 5 & 5 \\ 0 & -5 & 5 & 5 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{-5} & 5 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U.$$ Hence $\operatorname{rank} M = 2$ and $\dim \operatorname{null} A = 2$.

Solving $Uv = \vec{0}$ with $v_3 = 1$, $v_4 = 0$ gives $v = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$. Solving $Uv = \vec{0}$ with $v_3 = 0$, $v_4 = 1$ gives $v = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

(c) The matrix A has only two different eigenvalues: the eigenvalue from (a), and $\lambda = -3$. Is the matrix A diagonalizable? Explain!

There are altogether three linearly independent eigenvectors: $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. Since we don't have $n = 4$ linearly independent eigenvectors the matrix A is NOT diagonalizable.

5. Find all eigenvalues and eigenvectors for the matrices

$$\text{(i)} \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}, \qquad \text{(ii)} \begin{bmatrix} 1 & 2 \\ -5 & 3 \end{bmatrix}, \qquad \text{(iii)} \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}, \qquad \text{(iv)}\, A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Which of these matrices are diagonalizable?

(i): $p(\lambda) = \lambda^2 - 3$, $\lambda_1 = \sqrt{3}$, $v^{(1)} = \begin{bmatrix} 1 + \sqrt{3} \\ 1 \end{bmatrix}$, $\lambda_2 = -\sqrt{3}$, $v^{(2)} = \begin{bmatrix} 1 - \sqrt{3} \\ 1 \end{bmatrix}$

(ii): $p(\lambda) = \lambda^2 - 4\lambda + 13$, $\lambda_1 = 2 + 3i$, $v^{(1)} = \begin{bmatrix} 2 \\ 1 + 3i \end{bmatrix}$, $\lambda_1 = 2 - 3i$, $v^{(1)} = \begin{bmatrix} 2 \\ 1 - 3i \end{bmatrix}$

(iii): $p(\lambda) = \lambda^2 - 4\lambda + 4$, $\lambda_1 = \lambda_2 = 2$, $v^{(1)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (only one eigenvector exists!)

(iv): $p(\lambda) = (3 - \lambda)^2$, $\lambda_1 = \lambda_2 = 3$, $v^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (or any other basis of $\mathbb{C}^2$)

The matrices (i), (ii), (iv) are diagonalizable. The matrix (iii) is NOT diagonalizable.