Solution Exam #2

1. **(a)** We obtain $P^\top P = \begin{bmatrix} 10 & 0 \\ 0 & 4 \end{bmatrix}$. This means that the columns $p^{(1)}, p^{(2)}$ satisfy $p^{(1)} \cdot p^{(2)} = 0$, i.e., the columns are orthogonal on each other.

(b) With $S = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ we have $A = PS$, hence $\|Ac - y\| = \|P \underbrace{Sc}_{d} - y\| = \min$. The normal equations for this are $P^\top P d = P^\top y$ or $\begin{bmatrix} 10 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$ yielding $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}$. Solving $Sc = d$ or $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}$ gives $c_2 = -\frac{1}{2}$, $c_1 = 1$, i.e., $c = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$.

2. We have $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $y = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and need to solve $\|Ac - y\| = \min$. The normal equations are $A^\top A c = A^\top y$ or $\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$, yielding $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix}$. Then $r = Ac - y = \begin{bmatrix} \frac{3}{2} \\ 0 \\ -\frac{3}{2} \end{bmatrix}$ and $\|r\| = \frac{3}{2}\sqrt{2}$.

3. **(a)** Using the expansion formula with the first row gives

$$\det \begin{bmatrix} 4-\lambda & 0 & 0 \\ -3 & 1-\lambda & 3 \\ 3 & 3 & 1-\lambda \end{bmatrix} = (4-\lambda) \det \begin{bmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{bmatrix} = (4-\lambda)(\lambda^2 - 2\lambda - 8)$$

The equation $\lambda^2 - 2\lambda - 8 = 0$ has the solutions $-2, 4$. Hence the eigenvalues are $4, 4, -2$.

(b) Find the eigenvectors: For $\lambda = 4$ we have to solve $\begin{bmatrix} 0 & 0 & 0 \\ -3 & -3 & 3 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The matrix has rank 1, hence there are two eigenvectors, e.g., $v^{(1)} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $v^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. For $\lambda = -2$ we have to solve $\begin{bmatrix} 6 & 0 & 0 \\ -3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The matrix has rank 2, hence there is one eigenvector $v^{(3)} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. Therefore we have $V = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. Note that other choices are possible for the eigenvectors; we can also use a different order of the eigenvalues.

4. For $\lambda = 2$ we have $M = A - \lambda I = \begin{bmatrix} 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 2 & -4 \end{bmatrix}$ giving the row echelon form $\begin{bmatrix} 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
Hence the rank of M is 2, and the dimension of the eigenspace is $4 - 2 = 2$.

For $\lambda = -3$ we have $M = A - \lambda I = \begin{bmatrix} 5 & 1 & -1 & 1 \\ 0 & 5 & 1 & -2 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$ giving the row echelon form $\begin{bmatrix} 5 & 1 & -1 & 1 \\ 0 & 5 & 1 & -2 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
Hence the rank of M is 3, and the dimension of the eigenspace is $4 - 3 = 1$.
Hence the total number of linearly independent eigenvectors is $2 + 1 = 3$. Since this is less than 4, the matrix is NOT diagonizable.