1 Introduction: scalars, vectors, matrices

1.1 Scalars and vectors

A **scalar** is a real number $c \in \mathbb{R}$.

A **vector** $\vec{u} \in \mathbb{R}^n$ has the form $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ with components $u_1, \ldots, u_n \in \mathbb{R}$.

(Later we will also consider complex scalars $c \in \mathbb{C}$ and complex vectors $\vec{u} \in \mathbb{C}^n$.)

The zero vector is $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.

Geometric interpretation: A vector $\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ can represent

- either the **point** in $\mathbb{R}^n$ with coordinates $u_1, \ldots, u_n$.
- or an **arrow** in $\mathbb{R}^n$

  The starting point may be any point $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$, the end point is $\begin{bmatrix} x_1 + u_1 \\ \vdots \\ x_n + u_n \end{bmatrix}$

Basic operations: For vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$and a scalar $c \in \mathbb{R}$ we define the following operations:

vector+vector: $\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}$

scalar·vector: $c \cdot \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} c \cdot u_1 \\ \vdots \\ c \cdot u_n \end{bmatrix}$

Example: Given two points $\vec{u}, \vec{v} \in \mathbb{R}^n$, find the midpoint $\vec{m}$.
The vector $\vec{d} := \vec{v} - \vec{u}$ points from the point $\vec{u}$ to the point $\vec{v}$ so that we have $\vec{v} = \vec{u} + \vec{d}$. We can obtain the midpoint by adding $\frac{1}{2}\vec{d}$ to $\vec{u}$:

$$\vec{m} = \vec{u} + \tfrac{1}{2}\vec{d} = \vec{u} + \tfrac{1}{2}(\vec{v} - \vec{u}) = \tfrac{1}{2}\vec{u} + \tfrac{1}{2}\vec{v}$$

Example: Given two points $\vec{u}, \vec{v} \in \mathbb{R}^n$, find all points on the line through $\vec{u}$ and $\vec{v}$:

Let $t \in \mathbb{R}$, then adding $t \cdot \vec{d}$ to $\vec{u}$ gives a point on the line:

$$\vec{u} + t \cdot \vec{d} = \vec{u} + t(\vec{v} - \vec{u}) = (1-t)\vec{u} + t\vec{v}$$

Note that

- $t = 0$ gives the point $\vec{u}$
  $t = 1$ gives the point $\vec{v}$
- $0 < t < 1$ gives a point between $\vec{u}$ and $\vec{v}$
- $t < 0$ gives points on the line beyond the point $\vec{u}$
- $t > 1$ gives points on the line beyond the point $\vec{v}$

1.2 Linear combinations, spans, linear independence

We call

$$c_1\vec{u}^{(1)} + \cdots + c_k\vec{u}^{(k)}$$

a **linear combination** of the vectors $\vec{u}^{(1)},\ldots,\vec{u}^{(k)} \in \mathbb{R}^n$ with scalars $c_1,\ldots,c_k \in \mathbb{R}$.

The set of all possible linear combinations is called the **span** of the vectors $\vec{u}^{(1)},\ldots,\vec{u}^{(k)}$:

$$\operatorname{span}\left\{\vec{u}^{(1)},\ldots,\vec{u}^{(k)}\right\} = \left\{c_1\vec{u}^{(1)} + \cdots + c_k\vec{u}^{(k)} \mid c_1,\ldots,c_k \in \mathbb{R}\right\}$$

Examples:

1. $\operatorname{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$ is the line through the origin and the point $\begin{bmatrix}1\\1\\0\end{bmatrix}$

2. $\operatorname{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$ is the x_1x_2 plane

3. $\operatorname{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ is also the x_1x_2 plane

4. $\operatorname{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ is the whole space $\mathbb{R}^3$

5. $\operatorname{span}\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\} = \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$ is just the origin

We see that spans in $\mathbb{R}^3$ are one of the following:

- just the origin $\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$
- lines through the origin
- planes through the origin
- the whole space $\mathbb{R}^3$

We call these sets **subspaces** of $\mathbb{R}^3$. In general a **subspace** of $\mathbb{R}^n$ is a subset $U \subset \mathbb{R}^n$ such that

- $\vec{u} \in U$ implies $c \cdot \vec{u} \in U$ for any $c \in \mathbb{R}$
- $\vec{u},\vec{v} \in U$ implies $\vec{u}+\vec{v} \in U$

The first property with $c = 0$ shows that U must contain the origin $\vec{0}$.

In example 3. we considered $\operatorname{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$. In this case one the of the vectors is redundant: We can e.g. express the third vector as a linear combination of the other two vectors:

$$\begin{bmatrix}0\\1\\0\end{bmatrix} = 1\cdot\begin{bmatrix}1\\1\\0\end{bmatrix} + (-1)\cdot\begin{bmatrix}1\\0\\0\end{bmatrix}$$

or by moving everything to the left hand side

$$1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

i.e., we have a linear combination $c_1\vec{u}^{(1)} + c_2\vec{u}^{(2)} + c_3\vec{u}^{(3)} = \vec{0}$, but not all of the coefficients c_1, c_2, c_3 are zero.

In such a case we say that the vectors $\vec{u}^{(1)}, \vec{u}^{(2)}, \vec{u}^{(3)}$ are **linearly dependent**.

Definition 1. We say that vectors $\vec{u}^{(1)}, \ldots, \vec{u}^{(k)} \in \mathbb{R}^n$ are **linearly dependent** if there are coefficents $c_1, \ldots, c_k$ which are not all zero such that

$$c_1\vec{u}^{(1)} + \cdots + c_k\vec{u}^{(k)} = \vec{0}.$$

If e.g. $c_2 \neq 0$ this means we can write $\vec{u}^{(2)}$ as a linear combination of the remaining vectors. Hence we have

- The vectors $\vec{u}^{(1)}, \ldots, \vec{u}^{(k)} \in \mathbb{R}^n$ are **linearly dependent** if we can write one of the vectors as a linear combination of the remaining ones
- The vectors $\vec{u}^{(1)}, \ldots, \vec{u}^{(k)} \in \mathbb{R}^n$ are **linearly independent** if

  $$c_1\vec{u}^{(1)} + \cdots + c_k\vec{u}^{(k)} = \vec{0}$$

  can only happen for $c_1 = \cdots = c_k = 0$.

Example: Show that the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ in example 4. are linearly independent:

Consider a linear combination

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_1 + c_2 + c_3 \\ c_3 \end{bmatrix}$$

If this is the zero vector we have $\begin{bmatrix} c_1 \\ c_1 + c_2 + c_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, i.e., we have the three equations

$$c_1 = 0, \qquad c_1 + c_2 + c_3 = 0, \qquad c_3 = 0$$

which imply $c_1 = c_2 = c_3 = 0$. Hence the three vectors are linearly independent.

Consider a subspace U given by

$$U = \operatorname{span}\{\vec{u}^{(1)}, \ldots, \vec{u}^{(k)}\}$$

- if the vectors $\vec{u}^{(1)}, \ldots, \vec{u}^{(k)}$ are linearly dependent, we can remove some of them and still obtain the same subspace U
- if the vectors $\vec{u}^{(1)}, \ldots, \vec{u}^{(k)}$ are linearly independent we cannot remove any of them. In this case we say that the vectors $\vec{u}^{(1)}, \ldots, \vec{u}^{(k)}$ form a **basis of the subspace U**.

Note that there a many possible choices for a basis of a subspace U: In example 4. we saw that

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \text{is a basis of } U = \mathbb{R}^3$$

But we could also choose the following:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{is a basis of } U = \mathbb{R}^3$$

1.3 Matrices

A matrix $A \in \mathbb{R}^{m \times n}$ is a rectangular array with m rows and n columns

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

containing entries $a_{ij} \in \mathbb{R}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

For a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $\vec{x} \in \mathbb{R}^n$ we define the **matrix-vector product** $\vec{y} = A\vec{x} \in \mathbb{R}^m$ by

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = A\vec{x} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix}$$

("multiply rows by columns").

Note that a matrix $A \in \mathbb{R}^{m \times n}$ maps a vector $\vec{x} \in \mathbb{R}^n$ to a vector $\vec{y} \in \mathbb{R}^m$.

Example:

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 + 1 \cdot 3 \\ (-1) \cdot 2 + 3 \cdot 3 \\ 1 \cdot 2 + 4 \cdot 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 9 \end{bmatrix},$$

i.e., the matrix maps the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ to the vector $\begin{bmatrix} 7 \\ 7 \\ 9 \end{bmatrix}$.

We can look at this also in the following way:

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + 3 \cdot \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix},$$

i.e., a matrix-vector product yields a **linear combination of the n columns of the matrix** A:

$$A\vec{x} = x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$