Linear Systems of m equations for n unknowns

Introduction

We want to find all solutions for a linear system of m equations for n unknowns.
Example with $m = 3$ and $n = 4$:

$$\begin{bmatrix} 0 & 0 & 1 & -2 \\ 2 & 4 & 2 & -2 \\ 1 & 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix}$$

Using Gaussian elimination with pivoting to find the row echelon form

Column 1: From the three pivot candidates in column 1 we select 2 as a pivot.
We interchange rows 1 and 2.
We perform elimination using the pivot 2, generating zeros in column 1.

$$L = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 1 & -2 \\ 2 & 4 & 2 & -2 \\ 1 & 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix}, \quad p = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$L = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}, \qquad \begin{bmatrix} 2 & 4 & 2 & -2 \\ 0 & 0 & 1 & -2 \\ 1 & 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 8 \end{bmatrix}, \quad p = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$L = \begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot \\ \frac{1}{2} & \cdot & \cdot \end{bmatrix}, \qquad \begin{bmatrix} 2 & 4 & 2 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix}, \quad p = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Column 2: We have two pivot candidates **which are all zero**. We therefore move on to the next column.

$$L = \begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot \\ \frac{1}{2} & \cdot & \cdot \end{bmatrix}, \qquad \begin{bmatrix} 2 & 4 & 2 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix}, \quad p = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Column 3: From the two pivot candidates in column 3 we select 1 as a pivot.
We perform elimination using the pivot 1, generating a zero in column 3.

$$L = \begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot \\ \frac{1}{2} & \cdot & \cdot \end{bmatrix}, \qquad \begin{bmatrix} 2 & 4 & 2 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix}, \quad p = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$L = \begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot \\ \frac{1}{2} & 2 & \cdot \end{bmatrix}, \qquad \begin{bmatrix} 2 & 4 & 2 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}, \quad p = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Result: From the matrix A we obtained

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 2 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 2 & 4 & 2 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad p = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

such that $LU = \begin{bmatrix} \text{row } p_1 \text{ of } A \\ \vdots \\ \text{row } p_m \text{ of } A \end{bmatrix}$.

We see that the **rank of the matrix** A is $r = 2$ since U has two nonzero rows. The first nonzero element in rows $1, \ldots, r$ is the **pivot**. Here the pivots are in columns 1 and 3. Hence x_1 and x_3 are **basic variables**, the remaining variables x_2 and x_4 are **free variables**.

We also obtained from the original right hand side vector $b = \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix}$ the new right hand side vector $y = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$. Note that the original and the new linear systems are equivalent and have the same solutions:

$$Ax = b \iff Ux = y$$

We can obtain the new right hand side vector y in two ways:

- transform the vector b along with the matrix A, applying the same row interchanges and elimination operations.
- after obaining L, U, p from the matrix A solve the linear system $Ly = \begin{bmatrix} b_{p_1} \\ \vdots \\ b_{p_m} \end{bmatrix}$ using forward substitution.

Method 1 for solving the linear system $Ax = b$

This linear system is equivalent to $Ux = y$.

If $\begin{bmatrix} y_{r+1} \\ \vdots \\ y_m \end{bmatrix} \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$: There is no solution. STOP.

Otherwise: Use arbitrary values for the $n - r$ free variables. Determine the r basic variables by back substitutation from (eq.r),...,(eq 1).

In our example we have $r = 2$. Note that $y_3 = 0$, hence there exists a solution. The free variables are x_2, x_4, the basic variables are x_1, x_3.

We use (eq.2) to determine x_3:

$$x_3 - 2x_4 = 3 \qquad \iff \qquad x_3 = 3 + 2x_4$$

We use (eq.1) to determine x_1:

$$2x_1 + 4x_2 + 2x_3 - 2x_4 = 4 \qquad \iff \qquad x_1 = \frac{4 - 4x_2 - 2x_3 + 2x_4}{2} = \frac{4 - 4x_2 - 2(3 + 2x_4) + 2x_4}{2} = -1 - 2x_2 - x_4$$

Hence any solution $x \in \mathbb{R}^4$ of $Ax = b$ is given by

$$x = \begin{bmatrix} -1 - 2x_2 - x_4 \\ x_2 \\ 3 + 2x_4 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 0 \\ 3 \\ 0 \end{bmatrix}}_{x_{\text{part}}} + x_2 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{v^{(1)}} + x_4 \underbrace{\begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}}_{v^{(2)}}, \qquad x_2, x_4 \in \mathbb{R} \text{ arbitrary}$$

$$x = x_{\text{part}} + x_2 v^{(1)} + x_4 v^{(2)}, \qquad x_2, x_4 \in \mathbb{R} \text{ arbitrary}$$

- for the choice $x_2 = x_4 = 0$ we obtain a "particular solution" x_{part} satisfying $Ax_{\text{part}} = b$
- the vectors $v^{(1)}, v^{(2)}$ satisfy $Av^{(1)} = \vec{0}$, $Av^{(2)} = \vec{0}$
- for the right hand side vector $b = \vec{0}$ we obtain $Ax = \vec{0} \iff Ux = \vec{0}$ yielding $x = x_2 v^{(1)} + x_4 v^{(2)}$ with $x_2, x_4 \in \mathbb{R}$ arbitrary. Hence $\text{null} A = \text{span}\{v^{(1)}, v^{(2)}\}$.

Finding a basis of the null space

We can obtain the vectors $v^{(k)}$ satisfying $Av^{(k)} = \vec{0}$ for $k = 1, \ldots, n-r$ as follows:
For each free variable x_j do the following:

- Let $x_j := 1$, set all other free variables to zero.
- Find the basic variables by solving $Ux = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ with back substitution.

  This yields a solution vector

$$v = \begin{bmatrix} * \\ \vdots \\ * \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{c} \scriptstyle 1 \\ \vdots \\ \scriptstyle j-1 \\ \scriptstyle j \\ \scriptstyle j+1 \\ \vdots \\ \scriptstyle n \end{array} \tag{1}$$

.

Then $Av = \vec{0}$ means $v_1 a^{(1)} + \cdots + v_n a^{(n)}$ where $a^{(1)}, \ldots, a^{(n)}$ are the columns of A. Hence for this specific vector v we obtain

$$a^{(j)} = -v_1 a^{(1)} - \cdots - v_{j-1} a^{(j-1)}$$

i.e., we have written column j of the matrix as a linear combination of columns 1 through $j-1$.

Claim: The vectors $v^{(1)}, \ldots, v^{(n-r)}$ form a basis for null$A$.

Gaussian elimination gives $Ax = \vec{0} \iff Ux = \vec{0}$. Then back substitution yields $x = c_1 v^{(1)} + c_2 v^{(2)}$ with $c_1, c_2 \in \mathbb{R}$ arbitrary. Hence we have null$A = \text{span}\{v^{(1)}, \ldots, v^{(n-r)}\}$. It remains to show that the vectors $v^{(1)}, \ldots, v^{(n-r)}$ are linearly independent: Assume that

$$c_1 v^{(1)} + \cdots + c_{n-r} v^{(n-r)} = \vec{0} \tag{2}$$

Let j be the index of the last free variable. Then $v^{(n-r)}$ has the form (1), and the vectors $v^{(1)}, \ldots, v^{(n-r-1)}$ have the j-th component zero. Therefore (2) implies that $c_{n-r} = 0$. We can now repeat the same argument with the second to last free variable and $v^{(n-r-1)}$ and obtain $c_{n-r-1} = 0, \ldots,$ and $c_1 = 0$.

Method 2 for solving the linear system $Ax = b$

We can obtain the same answer as in Method 1 in the following way:

- Set all free variables to zero. Use back substitution for $Ux = y$ to find the basic variables. This yields a **particular solution** x_{part}.
- Find a basis $v^{(1)}, \ldots, v^{(n-r)}$ of the null space as described above.
- The general solution is given by

$$x = x_{\text{part}} + c_1 v^{(1)} + \cdots + c_{n-r} v^{(n-r)}, \qquad c_1, \ldots, c_{n-r} \in \mathbb{R} \text{ arbitrary}$$

Using symbolic Matlab to solve a linear system

Note that

```
>> A=sym([0 0 1 -2; 2 4 2 -2; 1 2 3 -5]);
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LU decomposition with echelon matrix U

For a general $m \times n$ matrix A we can apply Gaussian elimination in the same way as explained before for square matrices. However, if at some point all pivot candidates are zero, we do not stop but instead move to the next column.

Algorithm: Gaussian Elimination yielding echelon matrix U Here we denote the elements of the matrix A by $A(i,j)$. We use Matlab notation for row and column ranges, i.e., $A(3,:)$ is row 3.

Input: matrix $A \in \mathbb{R}^{m\times n}$.
Output: matrix $L \in \mathbb{R}^{m\times m}$ (lower triangular), matrix $U \in \mathbb{R}^{m\times n}$ (row echelon form), vector $p \in \mathbb{R}^m$ (contains permutation of $1,\ldots,m$)

$$L := \underbrace{\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}}_{m\times m}; \quad U := A; \quad p := \begin{bmatrix} 1 \\ \vdots \\ m \end{bmatrix} \qquad \textit{(initializations)}$$

$c := 1$ *(current column)*
For $k = 1$ to m *(Find pivot for row k)*
  While $c \le n$ And $U(k:m,c) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ *(if **all** pivot candidates are zero)*
    $c := c+1$ *(go to next column)*
  End
  If $c > n$
    Break *(There is no pivot for row k, break out of For loop)*
  End
  Pick $j \in \{k,\ldots,m\}$ such that $U(j,c) \neq 0$ *(Pick a nonzero pivot candidate)*
  Interchange rows j and k of L, U, p
  For $l = k+1$ to m
    $L(l,k) := U(l,c)/U(k,c)$ *(compute multiplier)*
    $U(l,:) := U(l,:) - L(l,k)U(k,:)$ *(subtract multiplier times pivot row from current row)*
  End
  $c := c+1$ *(next column)*
End
For $k = 1$ to m
  $L(k,k) := 1$ *(Put 1's on the diagonal of L)*
End

The matrix U has **row echelon form**: Let r denote the number of nonzero rows of U. Rows $r+1,\ldots,m$ of U are zero. For $j = 1,\ldots,r$ we denote the column of the first nonzero element in row j by q_j, this element is called **pivot**. We have

$$1 \le q_1 < q_2 < \ldots < q_r \le n.$$

Columns $q_1,\ldots,q_r$ are called **pivot columns**.

We have

$$\boxed{LU = \tilde{A}} \qquad \text{where } \tilde{A} := \begin{bmatrix} \text{row } p_1 \text{of } A \\ \vdots \\ \text{row } p_m \text{of } A \end{bmatrix}$$

Let us define the $m \times m$ matrix $\hat{L}$ by

$$\begin{aligned} \text{row } p_1 \text{of } \hat{L} &:= \text{row } 1 \text{ of } L \\ &\vdots \\ \text{row } p_m \text{of } \hat{L} &:= \text{row } m \text{of } L \end{aligned}$$

then

$$\boxed{\hat{L}U = A}$$

How to use L, U, p for finding bases of ranges and nullspaces

Let $q_1, \ldots, q_r$ denote the pivot columns (corresponding to the basic variables).

Range of A:

Basis 1 is given by the columns $q_1, \ldots, q_r$ of the matrix A.

Basis 2 is given by the columns $1, \ldots, r$ of $\hat{L}$.

Nullspace of A:

For each of the $n-r$ free variables:

Set this free variable to 1, set all other free variables to 0. Then use $Ux = y$ and back substitution to find the basic variables: use equation r to find x_{q_r}, ..., use equation 1 to find x_{q_1}.

This gives a vector of the form $x = \begin{array}{c} \begin{bmatrix} * \\ \vdots \\ * \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \begin{array}{c} \scriptstyle 1 \\ \vdots \\ \scriptstyle j-1 \\ \scriptstyle j \\ \scriptstyle j+1 \\ \vdots \\ \scriptstyle n \end{array}$

Let us denote the $n-r$ vectors which we obtain in this way by $v^{(1)}, \ldots, v^{(n-r)}$. Then $\operatorname{null}(A) = \operatorname{span}\{v^{(1)}, \ldots, v^{(n-r)}\}$.

Range of $A^\top$:

Basis 1 is given by the rows $1, \ldots, r$ of U.

Basis 2 is given by the rows $p_1, \ldots, p_r$ of A, i.e. the rows of A where the pivot elements come from.

Nullspace of $A^\top$:

For $j = 1, \ldots, m-r$:

Solve $L^\top u = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{c} \scriptstyle 1 \\ \vdots \\ \\ \scriptstyle r+j \\ \\ \vdots \\ \scriptstyle m \end{array}$ using back substitution. Then let $\begin{cases} w_{p_1} := u_1 \\ \quad\vdots \\ w_{p_m} := u_m \end{cases}$

Let us denote the $m-r$ vectors which we obtain in this way by $w^{(1)}, \ldots, w^{(m-r)}$. Then $\operatorname{null}(A^\top) = \operatorname{span}\{w^{(1)}, \ldots, w^{(m-r)}\}$.

How to use L, U, p to solve a linear system $Ax = b$

- Solve $Ly = \begin{bmatrix} b_{p_1} \\ \vdots \\ b_{p_m} \end{bmatrix}$ using forward substitution.
- If $y_{r+1}, \ldots, y_m$ are not all zero: There is **no solution**.
- If $y_{r+1}, \ldots, y_m$ are all zero we can **find a particular solution** as follows:
  Set all free variables to zero. Then use $Ux = y$ and back substitution to find the basic variables: use equation r to find x_{q_r}, ..., use equation 1 to find x_{q_1}.
- The **general solution** is

$$x = x_{\text{part}} + c_1 v^{(1)} + \cdots + c_{n-r} v^{(n-r)} \quad \text{with } c_1, \ldots, c_{n-r} \in \mathbb{R} \text{arbitrary}$$

  where $v^{(1)}, \ldots, v^{(n-r)}$ is a basis of $\text{null}(A)$.

The following statements are equivalent:

- The linear system $Ax = b$ has a solution
- $b \in \text{range}(A)$
- $y_{r+1}, \ldots, y_m$ are all zero
- $w^{(j)\top} b = 0$ for $j = 1, \ldots, m - r$ where $w^{(1)}, \ldots, w^{(m-r)}$ is a basis of $\text{null}(A^\top)$.

Dot product (a.k.a. scalar product or inner product)

For two column vectors $u, v \in \mathbb{R}^n$ the expression

$$u^\top v = [u_1 \ldots u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + \cdots + u_n v_n$$

gives a scalar. This is the so-called **dot product** $\boxed{u \cdot v := u_1 v_1 + \cdots + u_n v_n}$.

Vectors u, v with $u \cdot v = 0$ are called **orthogonal**.

Orthogonal complement

If U is a subspace of $\mathbb{R}^n$ we can consider the vectors which are orthogonal on all vectors of U:

$$U^\perp := \{v \in \mathbb{R}^n \mid v \cdot u = 0 \text{ for all } u \in \mathbb{R}^n\}$$

This is again a subspace which is called the **orthogonal complement of** U.

Examples for $\mathbb{R}^3$: The 2-dimensional subspace $U = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a plane through the origin. A vector $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$ which is orthogonal on all vectors in U satisfies $v \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = v_1 = 0$ and $v \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = v_2 = 0$. Hence

$$U^\perp = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

which is a 1-dimensional subspace, i.e., a line through the origin.

The orthogonal complement of $V = \operatorname{span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ consists of all vectors $u \in \mathbb{R}^3$ with $u \cdot \begin{bmatrix}0\\0\\1\end{bmatrix} = u_3 = 0$, hence

$$V^\perp = \operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}.$$

$\operatorname{null} A$ and $\operatorname{range} A^\top$ are orthogonal complements

By definition the null space of A consists of all vectors x with $Ax = \vec{0}$, i.e., the vector x satisfies the m equations

$$\begin{bmatrix}a_{11}\\\vdots\\a_{1n}\end{bmatrix} \cdot \begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} = 0, \ \dots \ \begin{bmatrix}a_{m1}\\\vdots\\a_{mn}\end{bmatrix} \cdot \begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} = 0 \qquad (3)$$

By definition the range of $A^\top$ is the span of its columns:

$$A^\top = \operatorname{span}\left\{\begin{bmatrix}a_{11}\\\vdots\\a_{1n}\end{bmatrix}, \dots, \begin{bmatrix}a_{m1}\\\vdots\\a_{mn}\end{bmatrix}\right\}$$

Therefore the orthogonal complement of $A^\top$ consists of exactly the vectors $x \in \mathbb{R}^n$ satisfying (3).

$\operatorname{range} A$ and $\operatorname{null} A^\top$ are orthogonal complements

This follows by applying the above argument to $A^\top$ in place of A.

Solvability conditions for the linear system

Therefore we can rephrase the statement

$$\text{The linear system } Ax = b \text{ has a solution } \iff b \in \operatorname{range} A$$

as

$$\text{The linear system } Ax = b \text{has a solution } \iff b \text{ is orthogonal on all vectors in } \operatorname{null} A^\top$$

If we have a basis $w^{(1)}, \dots, w^{(m-r)}$ for $\operatorname{null} A^\top$ we obtain $m - r$ **solvability conditions:**

$$\textbf{The linear system } Ax = b \textbf{ has a solution } \iff \boxed{b \cdot w^{(1)} = 0, \dots, b \cdot w^{(m-r)} = 0} \qquad (4)$$

We can see from Gaussian elimination how to find these vectors $w^{(1)}, \dots, w^{(m-r)}$:

The linear system $Ax = b$ is equivalent to $Ux = y$ where $Ly = \tilde{b} := \begin{bmatrix}b_{p_1}\\\vdots\\b_{p_m}\end{bmatrix}$. The $m - r$ solvability conditions for $Ux = y$ are $y_{r+1} = 0, \dots, y_m = 0$.

The first condition can be written as $\begin{array}{c} \begin{array}{ccccccc} & & & {\scriptstyle r+1} & & & \end{array} \\ \left[\begin{array}{ccccccc} 0, & \cdots & ,0, & 1 & ,0, & \cdots & ,0 \end{array}\right] \end{array} \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = 0$. If we have a vector $u \in \mathbb{R}^m$ with

$$\begin{array}{c} \begin{array}{ccccccc} & & & {\scriptstyle r+1} & & & \end{array} \\ \left[\begin{array}{ccccccc} 0, & \cdots & ,0, & 1 & ,0, & \cdots & ,0 \end{array}\right] \end{array} = [u_1, \dots, u_m] L \tag{5}$$

we have

$$\begin{array}{c} \begin{array}{ccccccc} & & & {\scriptstyle r+1} & & & \end{array} \\ \left[\begin{array}{ccccccc} 0, & \cdots & ,0, & 1 & ,0, & \cdots & ,0 \end{array}\right] \end{array} \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = [u_1, \dots, u_m] \underbrace{Ly}_{\tilde{b}} = u \cdot \tilde{b}$$

i.e., the condition $y_{r+1} = 0$ is equivalent to $u \cdot \tilde{b} = 0$. By taking the transpose (5) becomes

$$L^\top u = \begin{array}{cc} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{c} {\scriptstyle 1} \\ \vdots \\ \\ {\scriptstyle r+1} \\ \\ \vdots \\ {\scriptstyle m} \end{array} \end{array}$$

By reordering the compoments of u we obtain a vector w such that $u \cdot \tilde{b} = w \cdot b$. This is exactly the first vector $w^{(1)}$ in the null space of $A^\top$ which we constructed above.

In this way we obtain vectors $w^{(1)}, \dots, w^{(m-r)}$ such that (4) holds.

Summary

The matrix A maps $\mathbb{R}^n$ to $\mathbb{R}^m$. We obtain *orthogonal decompositions* of $\mathbb{R}^n$ and $\mathbb{R}^m$ with the following dimensions and mapping properties:

$$\begin{array}{ccccccc} {\scriptstyle n} & & {\scriptstyle n-r} & & {\scriptstyle r} & & \\ \mathbb{R}^n & = & \mathrm{null}(A) & \overset{\perp}{+} & \mathrm{range}(A^\top) & & \\ \downarrow & & \downarrow & & \downarrow {\scriptstyle \text{1 to 1}} & & \\ \mathbb{R}^m & = & \{\vec{0}\} & + & \mathrm{range}(A) & \overset{\perp}{+} & \mathrm{null}(A^\top) \\ {\scriptstyle m} & & & & {\scriptstyle r} & & {\scriptstyle m-r} \end{array}$$

The matrix $A^\top$ maps $\mathbb{R}^m$ to $\mathbb{R}^n$:

$$\begin{array}{ccccccc} {\scriptstyle n} & & {\scriptstyle n-r} & & {\scriptstyle r} & & \\ \mathbb{R}^n & = & \mathrm{null}(A) & \overset{\perp}{+} & \mathrm{range}(A^\top) & + & \{\vec{0}\} \\ \uparrow & & & & \uparrow {\scriptstyle \text{1 to 1}} & & \uparrow \\ \mathbb{R}^m & = & & & \mathrm{range}(A) & \overset{\perp}{+} & \mathrm{null}(A^\top) \\ {\scriptstyle m} & & & & {\scriptstyle r} & & {\scriptstyle m-r} \end{array}$$

Example 1

Consider the 4×5 matrix $A = \begin{bmatrix} 2 & -1 & 4 & 2 & 3 \\ -2 & 1 & 2 & 0 & 2 \\ 6 & -3 & 0 & 2 & -1 \\ 0 & 0 & 2 & -2 & 1 \end{bmatrix}$.

Perform Gaussian elimination to find L, U, p:

Column 1: Pivot selection: $\begin{bmatrix} 2 & -1 & 4 & 2 & 3 \\ -2 & 1 & 2 & 0 & 2 \\ 6 & -3 & 0 & 2 & -1 \\ 0 & 0 & 2 & -2 & 1 \end{bmatrix}$

Elimination: $L = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ 3 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \end{bmatrix}$, $U = \begin{bmatrix} 2 & -1 & 4 & 2 & 3 \\ 0 & 0 & 6 & 2 & 5 \\ 0 & 0 & -12 & -4 & -10 \\ 0 & 0 & 2 & -2 & 1 \end{bmatrix}$, $p = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

Column 2: Pivot selection: $\begin{bmatrix} 2 & -1 & 4 & 2 & 3 \\ 0 & 0 & 6 & 2 & 5 \\ 0 & 0 & -12 & -4 & -10 \\ 0 & 0 & 2 & -2 & 1 \end{bmatrix}$ all candidates are zero, proceed to next column

Column 3: Pivot selection: $\begin{bmatrix} 2 & -1 & 4 & 2 & 3 \\ 0 & 0 & 6 & 2 & 5 \\ 0 & 0 & -12 & -4 & -10 \\ 0 & 0 & 2 & -2 & 1 \end{bmatrix}$

Elimination: $L = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ 3 & -2 & \cdot & \cdot \\ 0 & \frac{1}{3} & \cdot & \cdot \end{bmatrix}$, $U = \begin{bmatrix} 2 & -1 & 4 & 2 & 3 \\ 0 & 0 & 6 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{8}{3} & -\frac{2}{3} \end{bmatrix}$, $p = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

Column 4: Pivot selection: $U = \begin{bmatrix} 2 & -1 & 4 & 2 & 3 \\ 0 & 0 & 6 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{8}{3} & -\frac{2}{3} \end{bmatrix}$

interchanging rows 3 and 4 gives

$L = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ 0 & \frac{1}{3} & \cdot & \cdot \\ 3 & -2 & \cdot & \cdot \end{bmatrix}$, $U = \begin{bmatrix} 2 & -1 & 4 & 2 & 3 \\ 0 & 0 & 6 & 2 & 5 \\ 0 & 0 & 0 & -\frac{8}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $p = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \end{bmatrix}$

Elimination: use $L_{43} = 0$, this does not change U.

Column 5: Pivot selection: $U = \begin{bmatrix} 2 & -1 & 4 & 2 & 3 \\ 0 & 0 & 6 & 2 & 5 \\ 0 & 0 & 0 & -\frac{8}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ all candidates are zero

Since we are already at last column we are done!

Result:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 1 & 0 \\ 3 & -2 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & -1 & 4 & 2 & 3 \\ 0 & 0 & 6 & 2 & 5 \\ 0 & 0 & 0 & -\frac{8}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad p = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \end{bmatrix}$$

The rank is $r = 3$. The numbers of the pivot columns are $q = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$. I.e., the basic variables are x_1, x_3, x_4, the free variables are x_2, x_5.

Finding a basis for $\operatorname{null} A$:

Set $x_2 = 1$, $x_5 = 0$. Then use $U\vec{x} = \vec{0}$ to find x_4, x_3, x_1 by back substitution, yielding $\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Set $x_5 = 1$, $x_2 = 0$. Then use $U\vec{x} = \vec{0}$ to find x_4, x_3, x_1 by back substitution, yielding $\begin{bmatrix} \frac{1}{4} \\ 0 \\ -\frac{3}{4} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$

Finding a basis for $\operatorname{range} A$:

Use columns $q_1, \ldots, q_r$ of the matrix A, i.e. columns $1, 3, 4$: $\begin{bmatrix} 2 \\ -2 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ -2 \end{bmatrix}$

Finding a basis for $\operatorname{range} A^\top$:

Use rows $1, \ldots, r$ of U: $\begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 6 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{8}{3} \\ -\frac{2}{3} \end{bmatrix}$

Finding a basis for $\operatorname{null} A^\top$:

Solve $L^\top u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ by back substitution gives $u = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$. Let $\begin{cases} w_{p_1} := u_1 \\ w_{p_2} := u_2 \\ w_{p_3} := u_3 \\ w_{p_4} := u_4 \end{cases}$, this gives $w = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$.

Hence the linear system $Ax = b$ has a solution if and only if the right hand side vector $b \in \mathbb{R}^4$ satisfies the condition $w \cdot b = 0$, i.e.,

$$-b_1 + 2b_2 + b_3 = 0.$$

Example 2

Consider the 3×2 matrix $A = \begin{bmatrix} 2 & -4 \\ -1 & 2 \\ 1 & -2 \end{bmatrix}$. The matrix maps a vector $x \in \mathbb{R}^2$ to a vector $b = Ax \in \mathbb{R}^3$.

Gaussian elimination gives

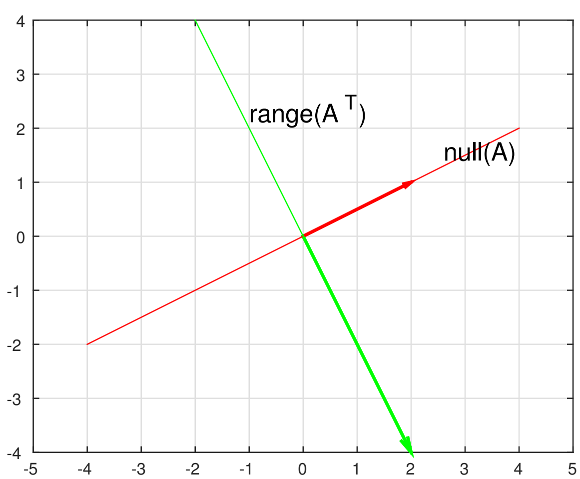
$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 2 & -4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad p = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

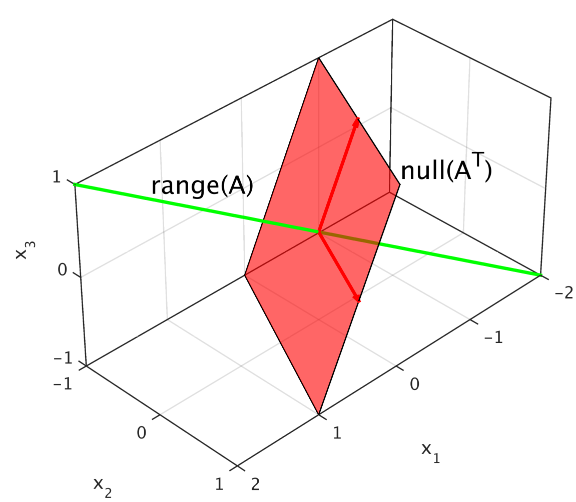
We see that the rank is $r = 1$ and $q_1 = 1$: the basic variable is x_1, the free variable is x_2. We obtain

- a basis for null$A$: solving $U \begin{bmatrix} x_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ gives $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
- a basis for range$A$: column q_1 of A is $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$
- a basis for range$A^\top$: row 1 of U is $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$
- a basis for null$A^\top$: solving $L^\top \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ gives $\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$, solving $L^\top \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ gives $\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$. Hence the basis is given by the vectors $\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$.

This shows how the matrix A maps a point in $\mathbb{R}^2$ to a point in $\mathbb{R}^3$: All points on the line null(A) are mapped to $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The points on the line range$(A^\top)$ are mapped 1-to-1 to the points on the line range(A).

This shows how the matrix $A^\top$ maps a point in $\mathbb{R}^3$ to a point in $\mathbb{R}^2$: All points on the plane null$(A^\top)$ are mapped to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The points on the line range(A) are mapped 1-to-1 to the points on the line range$(A^\top)$.





Example 3

Question: Find a basis for the orthogonal complement of the subspace $W = \text{span}\left\{\begin{bmatrix}1\\2\\3\\4\end{bmatrix},\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\}$.

Let us define $A = \begin{bmatrix}1 & 2 & 3 & 4\\1 & 1 & 1 & 1\end{bmatrix}$, then we have that $W = \text{range}A^\top$. Hence we need to find a basis for $W^\perp = \text{null}A$.

Gaussian elimination gives the row echelon form $U = \begin{bmatrix}1 & 2 & 3 & 4\\ 0 & -1 & -2 & -3\end{bmatrix}$, hence x_1, x_2 are basic variables and x_3, x_4 are free variables.

First vector for null$A$: Let $x_3 = 1$, $x_4 = 0$, use back substitution for $Ux = \vec{0}$ to find x_2, x_1. This gives $v^{(1)} = \begin{bmatrix}1\\-2\\1\\0\end{bmatrix}$

Second vector for null$A$: Let $x_3 = 0$, $x_4 = 1$, use back substitution for $Ux = \vec{0}$ to find x_2, x_1. This gives $v^{(2)} = \begin{bmatrix}2\\-3\\0\\1\end{bmatrix}$

Answer: We have the following basis: $W^\perp = \text{span}\left\{\begin{bmatrix}1\\-2\\1\\0\end{bmatrix},\begin{bmatrix}2\\-3\\0\\1\end{bmatrix}\right\}$

In Matlab we can use the `null` command for *symbolic matrices* to find this:

```
>> A = sym([1 2 3 4; 1 1 1 1]);
>> V = null(A)
V =
[  1,  2]
[ -2, -3]
[  1,  0]
[  0,  1]
```

Note that the `null` command in Matlab also works for *numerical matrices*, but gives a different basis:

```
>> A = [1 2 3 4; 1 1 1 1];
>> V = null(A)
V =
   -0.4001   -0.3741
    0.2546    0.7970
    0.6910   -0.4717
   -0.5455    0.0488
```

This is just another possible basis for the same subspace. Actually, for *numerical matrices* the `null` command returns an *orthonormal basis*: the vectors are orthogonal on each other and have length 1, i.e., $V^\top V = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$:

```
>> V'*V
ans =
    1.0000   -0.0000
   -0.0000    1.0000
```