Practice problems for final exam

Problems for Jordan normal form and quadratic forms

1. Let $A = \begin{bmatrix} 0 & 4 & 0 \\ -1 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. Find the Jordan normal form B and the nonsingular matrix $V \in \mathbb{R}^{3\times 3}$ so that $AV = VB$.

The eigenvalues are $-2, -2, -2$. There are two eigenvectors. We get the Jordan chains $v^{(1)} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $v^{(1,1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $v^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, hence we can use $V = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

2. The matrix $A = \begin{bmatrix} 10 & 5 & 1 \\ 2 & 9 & 5 \\ -2 & -3 & 5 \end{bmatrix}$ has only the eigenvalue $\lambda = 8$. Find the Jordan normal form B and the nonsingular matrix $V \in R^{3\times 3}$ so that $AV = VB$.

There is one eigenvector $\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$, and we get the Jordan chain $v^{(1)} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$, $v^{(1,1)} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $v^{(1,2)} = \begin{bmatrix} \frac{3}{4} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$, hence we can use $V = \begin{bmatrix} 3 & -1 & \frac{3}{4} \\ -1 & 1 & -\frac{1}{2} \\ -1 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 1 & 0 \\ 0 & 8 & 1 \\ 0 & 0 & 8 \end{bmatrix}$.

3. For the quadratic form $q(x_1, x_2) = 3x_1^2 - 4x_1x_2$ find the symmetric matrix $A \in \mathbb{R}^{2\times 2}$ such that $q(x) = x^\top Ax$. Find the eigenvalues. Is A positive semidefinite? If not, find $x \neq \vec{0}$ such that $x^\top Ax < 0$.

We have $A = \begin{bmatrix} 3 & -2 \\ -2 & 0 \end{bmatrix}$ which has the eigenvalues $-1, 4$. Since there is a negative eigenvalue the matrix is not positive semidefinite. We can e.g. use the eigenvector for $\lambda = -1$: Let $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then $x^\top Ax = 3 - 8 = -5$.

4. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and consider the quadratic form $q(x) = x^\top Ax$. Find a nonsingular matrix $V \in \mathbb{R}^{3\times 3}$ such that $B = V^\top AV$ is diagonal. Is A positive definite?

Hint: You should get $p(\lambda) = -\lambda^3 + 5\lambda^2 - 4\lambda$.

We obtain the eigenvalues $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 4$. Since we have a zero eigenvalue, the matrix is not positive definite. The eigenvectors are $v^{(1)} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $v^{(2)} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $v^{(3)} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Since the eigenvalues are different, these eigenvectors are orthogonal on each other. Dividing each eigenvector by its norm gives $V = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

Problems for Gaussian elimination and row echelon form

1. Let $A = \begin{bmatrix} 0 & -4 & 4 \\ -3 & -1 & -2 \\ -1 & 1 & -2 \\ 0 & -4 & 4 \end{bmatrix}$. Use Gaussian elimination to find the row echelon form U and L, p. Always use the first available pivot candidate in each column. What are the **dimensions** of of null(A), range(A), null($A^\top$), range($A^\top$)?

We obtain $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, $U = \begin{bmatrix} -3 & -1 & -2 \\ 0 & -4 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$, $p = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$. We have rank $r = 3 = \dim \text{range}A = \dim \text{range}A^\top$, $\dim \text{null}A = 0$, $\dim \text{null}A^\top = 1$.

2. Let $A = \begin{bmatrix} 2 & 2 & -3 & 4 \\ 2 & 2 & -1 & 2 \\ 4 & 4 & 6 & -4 \end{bmatrix}$. Use Gaussian elimination to find the row echelon form U and L, p. Always use the first available pivot candidate in each column. What are the **dimensions** of of null(A), range(A), null($A^\top$), range($A^\top$)?

We obtain $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 6 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & 2 & -3 & 4 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $p = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. We have rank $r = 2 = \dim\ \text{range}A = \dim\ \text{range}A^\top$, $\dim \text{null}A = 2$, $\dim \text{null}A^\top = 1$.

3. For some matrix A we obtain from Gaussian elimination $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 3 & 1 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $p = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix}$.

(a) Find a basis for null(A), range(A), null($A^\top$), range($A^\top$). Use the given L, U, p, do NOT try to find A or $A^\top$.

basis for range$A$: $\begin{bmatrix} 0 \\ 3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix}$, basis for null$A$: $\begin{bmatrix} \frac{1}{3} \\ 1 \\ 1 \end{bmatrix}$

basis for range$A^\top$: $\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$, basis for null$A^\top$: $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(b) Find the general solution of the linear system $Ax = \begin{bmatrix} -2 \\ 5 \\ 3 \\ -2 \end{bmatrix}$.

solving $Ly = \begin{bmatrix} b_{p_1} \\ b_{p_2} \\ b_{p_3} \\ b_{p_4} \end{bmatrix}$ gives $y = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \end{bmatrix}$. Solve $Ux = y$ with the free variable $x_3 = 0$ gives $x_{\text{part}} = \begin{bmatrix} \frac{4}{3} \\ -1 \\ 0 \end{bmatrix}$. The general solution is therefore $x = \begin{bmatrix} \frac{4}{3} \\ -1 \\ 0 \end{bmatrix} + c \begin{bmatrix} \frac{1}{3} \\ 1 \\ 1 \end{bmatrix}$, $c \in \mathbb{R}$ arbitrary.