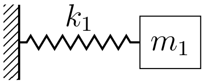
Application example: masses and springs

Introduction: one mass and one spring

We consider a mass m_1 and a spring with spring constant k_1. The mass can move horizontally. This is the picture at equilibrium:



Let $x_1(t)$ denote the displacement of the mass from the original position at time t. Assume that we pull with a horizontal force F_1 at the mass.

Newton's law states that

$$\text{mass} \cdot \text{acceleration} = \text{sum of all forces acting on mass}$$

The velocity of the mass is $x_1'(t)$, the acceleration is $x_1''(t)$. We have two forces acting on the mass

- the force from the spring is by **Hooke's law** given by $-k_1 \cdot x_1(t)$ since the length of the spring changes by $x_1(t)$
- the force F_1

Hence Newton's law gives

$$m_1 x_1''(t) = -k_1 x_1(t) + F_1$$

For a given force F_1 we now want to find an **equilibrium solution** where the mass does not move, i.e., the function $x_1(t)$ is constant, hence $x_1'(t) = 0$, $x_1''(t) = 0$.

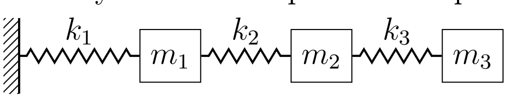
For an equilibrium solution we must therefore have a displacement x_1 satisfying

$$0 = -k_1 x_1 + F_1$$

and solving this gives $x_1 = F_1/k_1$.

Three masses and three springs

We consider three masses and three springs with spring constants k_1, k_2, k_3. The masses can move horizontally. This is the picture at equilibrium:



Let $x_1(t), x_2(t), x_3(t)$ denote the horizontal displacements of the three masses from their original positions at time t. Assume that we pull with horizontal forces F_1, F_2, F_3 at the three masses.

We now apply Newton's law for the mass m_1: The acceleration is $x_1''(t)$. There are three forces acting on the mass m_1:

- the force from spring 1 is given by $-k_1 x_1(t)$, since the length of the spring 1 changes by $x_1(t)$
- the force from spring 2 is given by $-k_2 (x_1(t) - x_2(t))$ since the length of spring 2 changes by $x_1(t) - x_2(t)$
- the force F_1

yielding

$$\begin{aligned} m_1 x_1''(t) &= -k_1 x_1(t) - k_2 (x_1(t) - x_2(t)) + F_1 \\ m_1 x_1''(t) &= -(k_1 + k_2) x_1(t) + k_2 x_2(t) + F_1 \end{aligned}$$

We can similarly apply Newton's law for mass m_2 and for mass m_3. We obtain the three equations

$$\begin{aligned} m_1 x_1''(t) &= -(k_1+k_2)x_1(t) + k_2 x_2(t) + F_1 \\ m_2 x_2''(t) &= k_1 x_1(t) - (k_2+k_3)x_2(t) + k_3 x_3(t) + F_2 \\ m_3 x_3''(t) &= k_3 x_2(t) - k_3 x_3(t) + F_3 \end{aligned}$$

or

$$\begin{bmatrix} m_1 x_1''(t) \\ m_2 x_2''(t) \\ m_3 x_3''(t) \end{bmatrix} + \underbrace{\begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

For a given forces F_1, F_2, F_3 we now want to find an **equilibrium solution** where the masses do not move, i.e., the functions $x_1(t), x_2(t), x_3(t)$ are constant, hence $x_1''(t) = x_2''(t) = x_3''(t) = 0$.

For an equilibrium solution we must therefore have a displacement vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying the linear system

$$A\vec{x} = \vec{F} \tag{1}$$

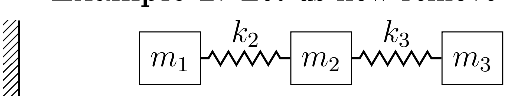
Solving this linear system gives all possible equilibrium solutions. If the linear system has no solution, then no equilibrium solution exists.

Example 1: Let $k_1 = 1$, $k_2 = 2$, $k_3 = 3$. In this case we obtain

$$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

Gaussian elimination gives $L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & -\frac{9}{11} & 1 \end{bmatrix}$, $U = \begin{bmatrix} 3 & -2 & 0 \\ 0 & \frac{11}{3} & -3 \\ 0 & 0 & \frac{6}{11} \end{bmatrix}$, $\vec{p} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. We see that $r = \operatorname{rank} A = 3$, hence $\dim \operatorname{null} A = 0$. Therefore the linear system (1) has for any right hand side vector $\vec{F}$ a unique solution $\vec{x}$.

Example 2: Let us now remove the spring 1, and let again $k_2 = 2$, $k_3 = 3$.



Removing the spring is the same as setting $k_1 = 0$ (then it does not cause any forces). Hence we have

$$A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

Gaussian elimination gives $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$, $\vec{p} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. We see that now $r = \operatorname{rank} A = 2$, hence $\dim \operatorname{null} A = 1$ and $\dim \operatorname{null} A^\top = 1$. Therefore the right hand side vector $\vec{F}$ must satisfy one condition for a solution to exist. If this condition is satisfied the general solution contains one free parameter.

Using U we can find a basis for $\operatorname{null} A$: Note that x_1, x_2 are basic variables, and x_3 is a free variable. Therefore we let $x_3 = 1$ and then use $U\vec{x} = \vec{0}$ to solve for x_2, x_1 by back substitution, yielding $x_2 = 1$, $x_1 = 1$, i.e.,

$$\operatorname{null} A = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\} \tag{2}$$

Using $L, \vec{p}$ we can find a basis for $\operatorname{null} A^\top$: Solving $L^\top \vec{u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ gives $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, i.e.,

$$\operatorname{null} A^\top = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}. \tag{3}$$

From (3) we obtain: The linear system has a solution only if $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \vec{F} = 0$, i.e.,

$$F_1 + F_2 + F_3 = 0. \tag{4}$$

If this condition is satisfied, the general solution has the form

$$\vec{x} = \vec{x}_{\text{part}} + c \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad c \in \mathbb{R} \text{ arbitrary} \tag{5}$$

because of (2).

This makes sense in terms of physics: Since spring 1 is no longer there, we can freely move the masses m_1, m_2, m_3 by the same amount which is expressed by (5). But if we want to have an equilibrium solution where the three masses are at rest, we now need that the the external forces acting on the three masses have a sum of zero which is expressed by (4). If the external forces have e.g. a positive sum, then the three masses together will keep accelerating toward the right.