Solution for Assignment #1, due February 15

1. Find the solution $u(x,t)$ of the following initial value problems: Find a formula for $u(x,t)$. Then use Matlab and `quasilin` to plot the characteristics and the solution.

(a) $u_t + (1-x) \cdot u_x = -u$, $u(x,0) = \exp(-x^2)$
solve IVP $X' = 1 - X$, $X(0) = x_0$: $X(t) = (x_0 - 1)e^{-t} + 1$
solve $x = (x_0 - 1)e^{-t} + 1$ for x_0: $x_0 = p(x,t) = (x-1)e^t + 1$
solve IVP $v' = -v$, $v(0) = u_0(x_0)$: $v(t) = u_0(x_0)e^{-t} = \exp(-x_0^2)e^{-t}$
$u(x,t) = \exp(-x_0^2)e^{-t}|_{x0=p(x,t)} = \exp\left[-\left((x-1)e^t+1\right)^2\right]e^{-t}$

(b) $u_t + t \cdot x \cdot u_x = -1$, $u(x,0) = \exp(-x^2)$
solve IVP $X' = t \cdot X$, $X(0) = x_0$: $X(t) = x_0 \exp(t^2/2)$
solve $x = x_0 \exp(t^2/2)$ for x_0: $x_0 = p(x,t) = x \cdot \exp(-t^2/2)$
solve IVP $v' = -1$, $v(0) = u_0(x_0)$: $v(t) = u_0(x_0) - t = \exp(-x_0^2) - t$
$u(x,t) = \exp(-x_0^2) - t|_{x0=p(x,t)} = \exp\left(-x^2\exp(-t^2)\right) - t$

(c) $u_t + (x+u) \cdot u_x = 0$, $u(x,0) = x$. *Hint:* First write down the ODE system for $X(t)$, $v(t)$.
solve IVP for the ODE system

$$\begin{aligned} X' &= X + v & \qquad X(0) &= x_0 \\ v' &= 0 & \qquad v(0) &= x_0 \end{aligned}$$

Here we can start with the second equation: $v' = 0$, $v(0) = x_0$ gives $v(t) = x_0$. Then we use this in the first equation: $X' = X + x_0$, $X(0) = x_0$ gives $X(t) = 2x_0e^t - x_0 = x_0(2e^t - 1)$
solve $x = x_0(2e^t - 1)$ for x_0: $x_0 = p(x,t) = x/(2e^t - 1)$
$u(x,t) = v(t)|_{x0=p(x,t)} = x_0|_{x0=p(x,t)} = x/(2e^t - 1)$

2. We consider traffic flow on a one-lane highway without exits. Let $u(x,t)$ denote the density. We assume that the maximal density is 1. The speed depends on the density: speed$= 1 - u$, i.e., at maximal density the speed is zero (traffic jam), at zero density the speed is 1 (maximal speed). The flux is given by $F(u) = \text{speed} \cdot u = (1-u)u$. Therefore we have the conservation law

$$u_t + \partial_x F(u) = 0, \qquad u(x,0) = u_0(x).$$

Here the initial density is given by

$$u_0(x) = \begin{cases} \frac{1}{3} & \text{for } x < 0 \\ \frac{1}{3} + \frac{2}{3}x & \text{for } 0 \le x < 1 \\ 1 & \text{for } x \ge 1 \end{cases}$$

(light traffic for $x < 0$, traffic jam for $x \ge 1$). We would like to know how the traffic jam changes with time.

(a) Try to find the solution with Matlab and `quasilin`. Then sketch the characteristics by hand. Find the time t_* which is the smallest time where characteristics cross.
We have $u_t + c(u)u_x = 0$ with $c(u) = F'(u) = 1 - 2u$. The characteristic through x_0 is given by $X_{(x0)}(t) = x_0 + c(u_0(x_0))t$.
For $x_0 < 0$ we have $u_0(x_0) = \frac{1}{3}$ and $X(t) = x_0 + \frac{1}{3}t$
For $0 < x_0 \le 1$ we have $u_0(x_0) = \frac{1}{3} - \frac{4}{3}x_0$ and $X(t) = x_0 + (\frac{1}{3} - \frac{4}{3}x_0)t$
For $x \le 1$ we have $u_0(x_0) = -1$ and $X(t) = x_0 - t$
The characteristic through $x_0 = 0$ is $X_{(0)}(t) = \frac{1}{3}t$. The characteristic through $x_0 = 1$ is $X_{(1)}(t) = 1 - t$. These two characteristics intersect at t_* with $X_{(0)}(t_*) = X_{(1)}(t_*)$, i.e.,

$$\tfrac{1}{3}t_* = 1 - t_* \iff t_* = \tfrac{3}{4},$$

i.e., they intersect in the point $(x_*, t_*) = (\frac{1}{4}, \frac{1}{3})$. Note that for $0 < x_0 \le 1$ we have $X_{(x_0)}(\frac{3}{4}) = x_0 + (\frac{1}{3} - \frac{4}{3}x_0)\frac{3}{4} = \frac{1}{4}$. So all of the characteristics with $0 \le x_0 \le 1$ pass through $(\frac{1}{4}, \frac{1}{3})$.

(b) For $t < t_*$ find a formula for the solution $u(x,t)$ (you need to distinguish three different cases depending on the location of (x,t)).

For $x < \frac{1}{3}t$ we have $u(x,t) = \frac{1}{3}$.
For $x \ge 1 - t$ we have $u(x,t) = 1$.
For $\frac{1}{3}t \le x \le 1 - t$ we have $u(x,t) = \frac{1}{3} + \frac{2}{3}p(x,t)$ where $p(x,t)$ is determined by solving $X_{(x_0)}(t) = x$ for x_0:

$$x_0 + (\tfrac{1}{3} - \tfrac{4}{3}x_0)t = x \implies x_0 = p(x,t) = \frac{x - \frac{1}{3}t}{1 - \frac{4}{3}t}, \qquad u(x,t) = \tfrac{1}{3} + \tfrac{2}{3}\frac{x - \frac{1}{3}t}{1 - \frac{4}{3}t} = \frac{2x - 2t + 1}{3 - 4t}$$

(c) Sketch the solution $u(x,t_*)$ for $t = t_*$. Try to construct for $t > t_*$ a shock which satisfies the Rankine-Hungoniot shock condition. Find a formula for the solution $u(x,t)$ for $t \ge t_*$ (you need to distinguish two cases depending on the location of (x,t). Sketch the characteristics (with the shock) for $0 \le t \le \frac{3}{2}$.
We will have a shock originating at (x_*, t_*). To the left of the shock we will have $u(x,t) = u_l = \frac{1}{3}$, to the right we will have $u(x,t) = u_r = 1$. The RH shock condition states that the speed a of the shock is with the flux $F(u) = (1-u)\cdot u$

$$a = \frac{F(u_r) - F(u_l)}{u_r - u_l} = \frac{0 \cdot 1 - \frac{2}{3}\cdot\frac{1}{3}}{1 - \frac{1}{3}} = -\frac{1}{3}$$

Therefore the shock has constant speed $-\frac{1}{3}$, and the location of the shock at time $t \ge t_*$ is given by $x = x_* - \frac{1}{3}(t - t_*) = \frac{1}{4} - \frac{1}{3}(t - \frac{3}{4}) = \frac{1}{2} - \frac{1}{3}t$. Therefore we have for $t > t_* = \frac{3}{4}$

$$u(x,t) = \begin{cases} \frac{1}{3} & \text{for } x < \frac{1}{2} - \frac{1}{3}t \\ 1 & \text{for } x > \frac{1}{2} - \frac{1}{3}t \end{cases}$$

The result is that end of the traffic jam moves to the left with speed $-\frac{1}{3}$. To the left of the shock we have light traffic with density $\frac{1}{3}$ and speed $\frac{2}{3}$, to the right of the shock we have a traffic jam with density 1 and speed 0.