Assignment #2: Solution

1.

(a) Setting $\det(A-\lambda I) = (6-\lambda)(9-\lambda) = 0$ gives the eigenvalues $\lambda_1 = 5$, $\lambda_2 = 10$. For $\lambda_1 = 5$ solving $(A-\lambda_1 I)\vec{v}^{(1)} = \vec{0}$ gives the eigenvector $\vec{v}^{(1)} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. For $\lambda_2 = 10$ we obtain the eigenvector $\vec{v}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Therefore the general solution of the ODE is

$$\vec{u}(t) = c_1 \vec{v}^{(1)} e^{-\lambda_1 t} + c_2 \vec{v}^{(2)} e^{-\lambda_2 t}$$

Using the initial condition $\vec{u}(0) = c_1\vec{v}^{(1)} + c_2\vec{v}^{(2)} = \vec{u}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ gives $c_1 = -\frac{2}{5}$, $c_2 = \frac{1}{5}$ and

$$\vec{u}^{(1)}(t) = -\tfrac{2}{5}\begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-5t} + \tfrac{1}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-10t} = \tfrac{1}{5}\begin{bmatrix} 4e^{-5t} + e^{-10t} \\ -2e^{-5t} + 2e^{-10t} \end{bmatrix}$$

Using the initial condition $\vec{u}(0) = c_1\vec{v}^{(1)} + c_2\vec{v}^{(2)} = \vec{u}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ gives $c_1 = \frac{1}{5}$, $c_2 = \frac{2}{5}$ and

$$\vec{u}^{(1)}(t) = \tfrac{1}{5}\begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-5t} + \tfrac{2}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-10t} = \tfrac{1}{5}\begin{bmatrix} -2e^{-5t} + 2e^{-10t} \\ e^{-5t} + 4e^{-10t} \end{bmatrix}$$

Hence we obtain

$$S(t) = \tfrac{1}{5}\begin{bmatrix} 4e^{-5t} + e^{-10t} & -2e^{-5t} + 2e^{-10t} \\ -2e^{-5t} + 2e^{-10t} & e^{-5t} + 4e^{-10t} \end{bmatrix}$$

(b) We have

$$\begin{aligned} \vec{u}(t) =& S(t)\vec{u}^{(0)} + \int_{s=0}^{t} S(t-s)\vec{f}(s)ds \\ u(\vec{t}) =& \tfrac{1}{5}\begin{bmatrix} 4e^{-5t} + e^{-10t} & -2e^{-5t} + 2e^{-10t} \\ -2e^{-5t} + 2e^{-10t} & e^{-5t} + 4e^{-10t} \end{bmatrix}\begin{bmatrix} u_1^{(0)} \\ u_2^{(0)} \end{bmatrix} \\ &+ \int_{s=0}^{t} \tfrac{1}{5}\begin{bmatrix} 4e^{-5(t-s)} + e^{-10(t-s)} & -2e^{-5(t-s)} + 2e^{-10(t-s)} \\ -2e^{-5(t-s)} + 2e^{-10(t-s)} & e^{-5(t-s)} + 4e^{-10(t-s)} \end{bmatrix}\begin{bmatrix} f_1(s) \\ f_2(s) \end{bmatrix} ds \end{aligned}$$

2. The solution formula gives using the definition of $u_0(y)$

$$u(x,t) = \int_{y=-\infty}^{\infty} S(x-y,t)u_0(y)dy = \int_{y=0}^{1} S(x-y,t)ydy + \int_{y=1}^{\infty} S(x-y,t)dy. \tag{1}$$

We first evaluate the second integral using $\partial_x U(x,t) = S(x,t)$: Hence $\partial_y\left[-U(x-y,t)\right] = S(x-y,t)$ and

$$\int_{y=1}^{\infty} S(x-y,t)dy = [-U(x-y,t]_{y=1}^{\infty} = \left[-g(t^{-1/2}(x-y))\right]_{y=1}^{\infty} = -\underbrace{\lim_{y\to-\infty} g(y)}_{0} + g(t^{-1/2}(x-y))$$

using $U(x-y,t) = g(t^{-1/2}x)$. For the first integral in (1) we use $\partial_x U(x,t) = S(x,t)$ and integration by parts:

$$\begin{aligned} \int_{y=0}^{1} S(x-y,t)ydy &= -\left[g(t^{-1/2}(x-y))y\right]_{y=0}^{1} + \int_{y=0}^{1} g(t^{-1/2}(x-y))dy = -g(t^{-1/2}(x-1)) - \left[t^{1/2}G(t^{-1/2}(x\right. \\ &= -g(t^{-1/2}(x-1)) - t^{1/2}G(t^{-1/2}(x-1)) + t^{1/2}G(t^{-1/2}x) \end{aligned}$$

yielding

$$\boxed{u(x,t) = -t^{1/2}G(t^{-1/2}(x-1)) + t^{1/2}G(t^{-1/2}x)}$$

3.

(a) Let $v_0(x)$ be the piecewise linear function with values $0, 1, 0$ at $x = -1, 0, 1$ and let $w_0(x) := 2v_0(x)$:

$$v_0(x) = \begin{cases} 0 & x < -1 \\ 1+x & -1 < x \le 0 \\ 1-x & 0 < x \le 1 \\ 0 & x > 1 \end{cases}$$

On the interval $[-1, 0]$ the function v_0 is a secant, the function w_0 is the tangent line at $x = -1$. Since $u_0'' < 0$ for $-1 < x < 0$ we must have $v_0(x) \le u_0(x) \le w_0(x)$. In the same way we can obtain the inequalities for $x \in [0, 1]$.
The functions u_0, v_0, w_0 are continuous and bounded, hence the corresponding solutions u, v, w of the heat equation are bounded classical solutions. The function $\tilde{u}(x,t) := v(x,t) - u(x,t)$ is also a bounded classical solution with initial data $\tilde{u}(x,0) = v_0(x) - u_0(x) \le 0$. By the maximum principle we must have $\tilde{u}(x,t) = v(x,t) - u(x,t) \le 0$ for all $x \in \mathbb{R}$, $t \ge 0$. The argument with $w(x,t)$ is analogous.

(b)
```
>> heat_pwlin([-1,0,1],[0,1,0],.7,.5)
ans =
      0.29411
>> heat_pwlin([-1,0,1],[0,2,0],.7,.5)
ans =
      0.58822
```

Note that we can obtain a good approximation for $u(x,t)$ by using a piecewise linear function with a finer grid:

```
x=-1:.01:1; heat_pwlin(x,1-x.^2,.7,.5)
ans =
         0.388422271734339
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4.

(a) We must have $u(x,0) = v(x)w(0) = \sin(2x)$, hence $v(x) = C\sin(2x)$. Let $C = 1$. Inserting $u(x,t) = \sin(2x)w(t)$ into the PDE gives $\sin(2x)w'(t) = -4\sin(2x)w(t)$, i.e., $w'(t) = -4w(t)$ and $w(0) = 1$. Therefore $w(t) = e^{-4t}$ and

$$u(x,t) = \sin(2x)e^{-4t}.$$

(b) Duhamel's principle gives with $u_{\text{hom}}(x,t) = \sin(2x)e^{-4t}$

$$u(x,t) = u_{\text{hom}}(x,t) + \int_{s=0}^{t} \int_{y=-\infty}^{\infty} S(x-y, t-s)s \, dy \, ds$$

Since $\int_{x=-\infty}^{\infty} S(x,t)dx = 1$ the integration with respect to y gives

$$u(x,t) = u_{\text{hom}}(x,t) + \int_{s=0}^{t} s \, ds = \boxed{\sin(2x)e^{-4t} + \tfrac{1}{2}t^2}$$