Assignment #3, due Wednesday, May 9

1. We consider the heat equation $u_t - 2u_{xx} = 0$ on the interval $[0,1]$ with boundary conditions

$$u(0,t) = 0, \qquad u_x(1,t) = 0$$

and initial condition $u(x,0) = 1$. In the series for the solution $u(x,t)$ find the **first two terms**.
Hint: $\int_0^{k\pi/2} \sin^2(z)dz = \frac{k\pi}{4}$ for integer k.
For the heat equation we first want to find special solutions of the form $g(t)v(x)$ with separated variables. Plugging this into the PDE and separating the variables gives

$$-\frac{v''(x)}{v(x)} = -\frac{g'(t)}{2g(t)} = \lambda$$

Solve eigenvalue problem $-v''(x) = \lambda v(x)$ on $[0,1]$ with boundary conditions $v(0) = 0$, $v'(1) = 0$: We have $\lambda > 0$ and

$$v(x) = C_1 \cos(\lambda^{1/2} x) + C_2 \sin(\lambda^{1/2} x)$$

Now $v(0) = 0$ gives $C_1 = 0$. We must have $C_2 \neq 0$, so $v'(1) = 0$ gives $\cos(\lambda^{1/2}) = 0$ which implies $\lambda^{1/2} = (j - \frac{1}{2})\pi$, $j = 1, 2, 3, \ldots$. Therefore we have the eigenvalues λ_j and eigenfunctions $v_j(x)$

$$\lambda_j = \left[(j - \tfrac{1}{2})\pi\right]^2, \qquad v_j(x) = \sin\left[(j - \tfrac{1}{2})\pi x\right], \qquad j = 1, 2, 3, \ldots$$

Now we solve the problem $-g'(t) = 2\lambda_j g(t)$ and obtain $g(t) = c_j e^{-2\lambda_j t}$. Therefore the special solutions are $c_j v_j(x) e^{-2\lambda_j t}$. The solution of the initial value problem is

$$u(x,t) = \sum_{j=1}^{\infty} c_j v_j(x) e^{-2\lambda_j t} \qquad c_j = \frac{\langle u_0, v_j \rangle}{\langle v_j, v_j \rangle}$$

where $u_0(x) = 1$. Here we have

$$\lambda_1 = \frac{1}{4}\pi^2, \qquad v_1(x) = \sin(\frac{\pi}{2}x), \qquad c_1 = \frac{\int_0^1 1 \cdot \sin(\pi/2)dx}{\int_0^1 \sin^2(\pi/2)dx} = \frac{2/\pi}{1/2} = \frac{4}{\pi}$$

$$\lambda_2 = \frac{9}{4}\pi^2, \qquad v_2(x) = \sin(\frac{3\pi}{2}x), \qquad c_2 = \frac{\int_0^1 1 \cdot \sin(\frac{3\pi}{2}x)dx}{\int_0^1 \sin^2(\frac{3\pi}{2}x)dx} = \frac{2/(3\pi)}{1/2} = \frac{4}{3\pi}$$

so that we have

$$u(x,t) = \frac{4}{\pi}\sin(\frac{\pi}{2}x)\exp(-\frac{\pi^2}{2}t) + \frac{4}{3\pi}\sin(\frac{3\pi}{2}x)\exp(-\frac{9\pi^2}{2}t) + R(x,t), \qquad |R(x,t)| \leq C\exp(-\frac{25\pi^2}{2}t).$$

2. We consider the wave equation $u_{tt} - 4u_{xx} = 0$ on the interval $[0,1]$ with boundary conditions

$$u(0,t) = 0, \qquad u'(1,t) = 0$$

and initial conditions $u(x,0) = u_0(x) = 0$, $u_t(x,0) = u_1(x) = 1$.

(a) Use the appropriate extension to define the function $\tilde{u}_1(x)$ for all $x \in \mathbb{R}$ and sketch the graph of this function. *Hint:* use an even extension at Neumann boundary, odd extension at Dirichlet boundary.
We obtain a function $\tilde{u}_1(x)$ on the real axis which is periodic with period 4 and

$$u_1(x) = \begin{cases} 1 & \text{for } x \in \cdots \cup [-4,-2] \cup [0,2] \cup [4,6] \cup \cdots \\ -1 & \text{for } x \in \cdots \cup [-2,0] \cup [2,4] \cup [6,8] \cup \cdots \end{cases}$$

(b) Write down the D'Alembert formula for the extended solution $\tilde{u}(x,t)$. Use this to find $u(\frac{1}{2},\frac{1}{2})$: mark the interval over which you have to integrate $\tilde{u}_1(x)$ on your graph of $\tilde{u}_1(x)$; then find the value of $u(\frac{1}{2},\frac{1}{2})$. Evaluate $u(x,\frac{1}{2})$ for $x \in [0,1]$.
Here $c = 2$, $u_0(x) = 0$, hence $\tilde{u}_0(x) = 0$ and the D'Alembert formula gives for $x \in [0,1]$

$$\begin{aligned}
\tilde{u}(x,t) &= \frac{1}{2c}\int_{y=x-ct}^{x+ct} \tilde{u}_1(y)dy = \frac{1}{4}\int_{x-2t}^{x+2t} \tilde{u}_1(y)dy \\
u(\tfrac{1}{2},\tfrac{1}{2}) &= \frac{1}{4}\int_{\frac{1}{2}-2(\frac{1}{2})}^{\frac{1}{2}+2(\frac{1}{2})} \tilde{u}_1(y)dy = \frac{1}{4}\int_{-\frac{1}{2}}^{\frac{3}{2}} \tilde{u}_1(y)dy = \frac{1}{4}\left(\int_{-\frac{1}{2}}^{0}(-1)dy + \int_0^{\frac{3}{2}} 1dy\right) = \frac{1}{4}\left(-\frac{1}{2}+\frac{3}{2}\right) = \frac{1}{4} \\
u(x,\tfrac{1}{2}) &= \frac{1}{4}\int_{x-2(\frac{1}{2})}^{x+2(\frac{1}{2})} \tilde{u}_1(y)dy = \frac{1}{4}\int_{x-1}^{x+1} \tilde{u}_1(y)dy = \frac{1}{4}\left(\int_{x-1}^{0}(-1)dy + \int_0^{x+1} 1dy\right) \\
&= \frac{1}{4}\left((x-1)+(x+1)\right) = \frac{x}{2}
\end{aligned}$$

3. Consider a square metal plate $G = [0,1]\times[0,1]$. At three sides it is cooled to temparature 0 (Dirichlet condition), at the remaining side it is insulated (Neumann condition). The temperature $u(x,y,t)$ satisfies the heat equation $u_t - 2\Delta u = 0$. We start with the initial temperature $u_0(x,y) = 1$. At what rate λ will the temperature decay, i.e., $|u(x,y,t)| \le ce^{-\lambda t}$? For large t give an approximation to $u(x,y,t)$. *Hint:* Find a solution of the form $e^{-\lambda t}v(x,y)$ with the smallest possible λ and find the coefficient C so that $u(x,y,t) = Ce^{-\lambda t}v(x,y) + \text{faster decaying terms}$.
Assume we have Dirichlet conditions on the left, bottom and top side of the square and Neumann conditions on the right side of the square:

$$u(0,y,t) = 0, \qquad u_x(1,y,t) = 0, \qquad u(x,0,t) = 0, \qquad u(x,1,t) = 0$$

Separation of variables: Find special solutions $v(x,y)g(t)$, plugging this into the PDE gives

$$-\frac{\Delta v(x,y)}{v(x,y)} = -\frac{g'(t)}{2g(t)} = \lambda$$

Solve eigenvalue problem $-v_{xx}(x,y) - v_{yy}(x,y) = \lambda v(x,y)$ on $[0,1]\times[0,1]$ with boundary conditions $v(0) = 0$, $v'(1) = 0$: Try to find eigenfunctions $v(x,y) = p(x)q(y)$ with separated variables. Plugging this into the eigenvalue equation gives

$$\underbrace{\frac{-p''(x)}{p(x)}}_{\mu} + \underbrace{\frac{-q''(y)}{q(y)}}_{\nu} = \lambda$$

with constants μ, $\tilde{\mu}$. This means for $p(x)$ and μ

$$-p''(x) = \mu p(x), \qquad p(0) = 0, \qquad p'(1) = 0$$

so we obtain from problem 1 that

$$\mu_j = \left[(j-\tfrac{1}{2})\pi\right]^2, \qquad p_j(x) = \sin\left[(j-\tfrac{1}{2})\pi x\right], \qquad j = 1,2,3,\ldots$$

We have for $q(y)$ and $\tilde{\mu}$ that

$$-q''(y) = \nu q(y), \qquad q(0) = 0, \qquad q(1) = 0$$

so we obtain

$$\nu_k = [k\pi]^2, \qquad q_k(y) = \sin[k\pi y], \qquad k = 1,2,3,\ldots$$

Therefore the eigenvalues λ_{jk} and eigenfunctions $v_{jk}(x,y)$ for the eigenvalue problem in the square are

$$\lambda_{jk} = \mu_j + \nu_k = \left[(j-\tfrac{1}{2})\pi\right]^2 + [k\pi]^2, \qquad v_{jk}(x,y) = \sin\left[(j-\tfrac{1}{2})\pi x\right]\sin[k\pi y], \qquad j = 1,2,\ldots, \quad k = 1,2,\ldots$$

The smallest eigenvalue is $\lambda_{11} = (\frac{\pi}{2})^2 + \pi^2 = \frac{5}{4}\pi^2$ with the eigenfunction $v_{11}(x,y) = \sin(\frac{\pi}{2}x)\sin(\pi y)$. As in problem 1 we obtain $g(t) = c_{jk}e^{-2\lambda_{jk}t}$. Therefore the solution of the initial value problem is

$$u(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{jk} v_{jk}(x,y) e^{-2\lambda_{jk}t}, \qquad c_{jk} = \frac{\langle u_0, v_{jk}\rangle}{\langle v_{jk}, v_{jk}\rangle}$$

Here

$$\langle u_0, v_{jk}\rangle = \langle 1, p_j(x)q_k(y)\rangle = \left(\int_0^1 p_j(x)dx\right)\left(\int_0^1 q_k(y)dy\right)$$
$$\langle v_{jk}, v_{jk}\rangle = \langle p_j(x)q_k(y), p_j(x)q_k(y)\rangle = \left(\int_0^1 p_j(x)dx\right)\left(\int_0^1 q_k(y)dy\right)$$

so we have

$$c_{11} = \frac{\left(\int_0^1 \sin(\frac{\pi}{2}x)dx\right)\left(\int_0^1 \sin(\pi y)dy\right)}{\left(\int_0^1 \sin^2(\frac{\pi}{2}x)dx\right)\left(\int_0^1 \sin^2(\pi y)dy\right)} = \frac{\left(\frac{2}{\pi}\right)\left(\frac{2}{\pi}\right)}{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} = \frac{16}{\pi^2}$$

and

$$u(x,y,t) = \frac{16}{\pi^2}\sin(\frac{\pi}{2}x)\sin(\pi y)\exp\left(-\frac{5\pi^2}{2}t\right) + R(x,y,t), \quad |R(x,y,t)| \le C'\exp(-\lambda_{21}t) = C'\exp\left(-\frac{13\pi^2}{2}t\right)$$

and $|u(x,y,t)| \le C\exp(-\lambda_{11}t) = C\exp(-\frac{5\pi^2}{2}t)$.

4. Consider a square membrane $G = [0,1]\times[0,1]$ which is fixed at three sides (Dirichlet conditions) and free at the remaining side (Neumann conditions). The displacement $u(x,y,t)$ satisfies the wave equation $u_{tt} - 4\Delta u = 0$. What is the lowest frequency ω which the membrane can generate? *Hint:* Find a solution of the form $u(x,y,t) = \cos(\omega t)v(x,y)$ with the smallest possible ω.
Separation of variables: Find special solutions $v(x,y)g(t)$. For $v(x,y)$ we obtain the same eigenvalue problem as in problem 3. For $g(t)$ we have

$$\frac{-g''(t)}{4g(t)} = \lambda_{jk}, \qquad -g''(t) = 4\lambda_{jk}g(t).$$

This ODE has the general solution

$$g(t) = A_{jk}\cos(\omega_{jk}t) + B_{jk}\sin(\omega_{jk}t) \qquad \omega_{jk} = 2\lambda_{jk}^{1/2}.$$

Therefore we can write the solution of the initial value problem as

$$u(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} v_{jk}(x,y)\left[A_{jk}\cos(\omega_{jk}t) + B_{jk}\sin(\omega_{jk}t)\right]$$

where the coefficients A_{jk} and $B_{jk}$are determined from the initial conditions $u_0(x,y)$ and $u_1(x,y)$. We see that the possible frequencies of the membrane are

$$\omega_{jk} = 2\lambda_{jk}^{1/2} = 2\left[(j-\tfrac{1}{2})^2 + k^2\right]^{1/2}\pi, \qquad j = 1,2,\ldots, \quad k = 1,2,\ldots$$

The lowest possible frequency of the membrane is therefore

$$\omega_{11} = 2\lambda_{11}^{1/2} = 2\left[\frac{5}{4}\right]^{1/2}\pi = \pi\sqrt{5}$$