

1 First order PDE and method of characteristics

A first order PDE is an equation which contains $u_x(x,t)$, $u_t(x,t)$ and $u(x,t)$. In order to obtain a unique solution we must impose an additional condition, e.g., the values of $u(x,t)$ on a certain line.

1.1 Linear 1st order PDE

A linear 1st order PDE is of the form

$$\tilde{a}(x,t)u_x + \tilde{b}(x,t)u_t + \tilde{c}(x,t)u = \tilde{g}(x,t).$$

We assume that $\tilde{c}(x,t) \neq 0$, then we can divide by $\tilde{c}(x,t)$ and obtain the PDE in the form

$$u_t(x,t) + c(x,t)u_x(x,t) + a(x,t)u(x,t) = g(x,t). \quad (1)$$

We want to consider t as “time”, and prescribe for $t = 0$ the initial condition

$$u(x,0) = u_0(x) \quad (2)$$

with a given function $u_0(x)$. The problem consisting of the PDE (1) and the initial condition (2) is called an **initial value problem**.

A **(classical) solution** of the initial value problem is a function $u(x,t)$ such that the derivatives $u_t(x,t)$, $u_x(x,t)$ exist, and $u(x,t)$, $u_t(x,t)$, $u_x(x,t)$ are all continuous. Furthermore the PDE (1) is satisfied for all points (x,t) , and the initial condition (2) is satisfied for all x .

1.2 Characteristics

We observe that $u_t(x,t) + c(x,t)u_x(x,t)$ is a directional derivative in the direction of the vector $(c(x,t), 1)$ in the (x,t) plane. If we plot all these direction vectors in the (x,t) plane we obtain a direction field. We can find curves $x = X(t)$ fitting into this direction field by solving the ODE

$$X'(t) = c(X(t), t). \quad (3)$$

These curves are called **characteristic curves** or **characteristics**. The characteristic passing through the point $(x_0, 0)$ on the x -axis satisfies the initial condition

$$X(0) = x_0.$$

We now consider a solution $u(x,t)$ of the initial value problem (1), (2) and the characteristic curve $x = X(t)$ with $X(0) = x_0$. Restricting $u(x,t)$ to the characteristic curve defines the function

$$v(t) := u(X(t), t).$$

We can find $v'(t)$ by using the chain rule

$$\begin{aligned} v'(t) &= u_x(X(t), t)X'(t) + u_t(X(t), t) \\ &= u_x(X(t), t)c(X(t), t) + u_t(X(t), t) \end{aligned}$$

using the ODE (3). Since $u(x,t)$ satisfies the PDE (1) the function $v(t)$ satisfies the ODE

$$v'(t) + a(X(t), t)v(t) = g(X(t), t) \quad (4)$$

and initial condition

$$v(0) = u_0(x_0). \quad (5)$$

The result is that we can solve the PDE by solving a family of 1st order ODEs: For a given point (x,t) we first have to find x_0 so that the corresponding characteristic $X(t)$ passes through (x,t) . We then solve the initial value problem (4), (5) for the solution v . Then $v(t)$ is the solution $u(x,t)$ in the point (x,t) .

1.3 Recipe for solving linear 1st order PDE

We obtain the following recipe for finding the solution $u(x,t)$ of the initial value problem (8), (5):

1. Find the characteristics: Solve the initial value problem

$$X'(t) = c(X(t), t), \quad X(0) = x_0.$$

Since the solution depends on x_0 we will denote it by $X_{(x_0)}(t)$.

2. For a given point (x,t) find the starting point x_0 of the characteristic through (x,t) : For given x,t solve the equation

$$X_{(x_0)}(t) = x$$

for x_0 , yielding an expression $x_0 = p(x,t)$.

3. With the function $X(t) = X_{(x_0)}(t)$ from step 1 solve the initial value problem

$$v'(t) + a(X(t), t)v(t) = g(X(t), t), \quad v(0) = u_0(x_0).$$

Since the solution depends on x_0 we will denote it by $v_{(x_0)}(t)$.

4. Now the solution of the initial value problem (8), (5) is obtained by setting $x_0 = p(x,t)$ with the function $p(x,t)$ from step 2:

$$u(x,t) = v_{(x_0)}(t) \Big|_{x_0=p(x,t)}$$

Example: Solve the initial value problem

$$u_t + xu_x + u = 3x, \quad u(x,0) = \tan^{-1}(x).$$

Here $c(x,t) = x$ and $u_0(x) = \tan^{-1}(x)$.

1. Find the characteristics: Solve the initial value problem

$$X'(t) = X(t), \quad X(0) = x_0.$$

The general solution of the ODE is $X(t) = Ce^t$, and the initial condition gives

$$X_{(x_0)}(t) = x_0 e^t.$$

2. For a given point (x,t) find the starting point x_0 of the characteristic through (x,t) : For given x,t solve the equation $X_{(x_0)}(t) = x$, i.e.

$$x_0 e^t = x$$

for x_0 : This gives $x_0 = xe^{-t}$, hence

$$p(x,t) = xe^{-t}.$$

3. With $X(t) = x_0 e^t$ solve the initial value problem $v'(t) + a(X(t), t)v(t) = g(X(t), t)$, $v(0) = u_0(x_0)$:

$$v'(t) + v(t) = 3x_0 e^t, \quad v(0) = \tan^{-1}(x_0).$$

This is a linear inhomogeneous ODE. The solution of the homogeneous ODE is Ce^{-t} . The particular solution is of the form ae^t , plugging this into the ODE yields $ae^t + ae^t = 3x_0 e^t$, hence $a = \frac{3}{2}x_0$. The general solution of the ODE is $v(t) = Ce^{-t} + \frac{3}{2}x_0 e^t$. The initial condition gives $C + \frac{3}{2}x_0 = \tan^{-1}x_0$ or $C = \tan^{-1}x_0 - \frac{3}{2}x_0$. Therefore we obtain

$$v_{(x_0)}(t) = \left[\tan^{-1}x_0 - \frac{3}{2}x_0 \right] e^{-t} + \frac{3}{2}x_0 e^t$$

4. Now we insert $x_0 = p(x,t) = xe^{-t}$ into this expression to obtain the solution:

$$\begin{aligned} u(x,t) &= v_{(x_0)}(t) \Big|_{x_0=xe^{-t}} = \left[\tan^{-1}(xe^{-t}) - \frac{3}{2}(xe^{-t}) \right] e^{-t} + \frac{3}{2}(xe^{-t})e^t \\ &= \tan^{-1}(xe^{-t})e^{-t} - \frac{3}{2}xe^{-2t} + \frac{3}{2}x \end{aligned}$$

1.4 Difficulties

We have reduced the solution of the PDE to the solution of two families of IVPs for ODEs. Does this method always give us a (classical) solution $u(x, t)$? What could possibly go wrong?

1.4.1 Interval where $X(t)$ exists

An ODE initial value problem $v'(t) = f(t, v(t))$, $v(0) = v_0$ has a unique solution $v(t)$ if the function f is sufficiently smooth (e.g., $f(t, v)$ and $f_v(t, v)$ are continuous). However, it can happen that the solution only exists on an interval $t \in [0, t_*)$ and it does not exist for $t \geq t_*$.

A simple example is the IVP

$$v'(t) = v(t)^2, \quad v(0) = v_0$$

with $v_0 > 0$. In this case the solution is $v(t) = \frac{1}{a^{-1} - t}$, and the solution exists only for $t < a^{-1}$.

For a given point (x_1, t_1) we need to follow the characteristic backwards to $t = 0$ in order to find $x_0 = p(x_1, t_1)$. This means we are solving the following ODE problem for decreasing t :

$$X'(t) = c(X(t), t), \quad X(t_1) = x_1$$

For the method of characteristics to work we need that the solution of this IVP exists for $t \in [0, t_1]$. If this is the case for all points (x_1, t_1) we are able to find the characteristic curve $(X(t), t)$ from (x_1, t_1) back to $(x_0, 0)$. We then need to solve the IVP for $v(t) := u(X(t), t)$. Note that this is a linear ODE, so the solution is guaranteed to exist for all times.

1.4.2 Smoothness of given function $u_0(x)$

For a classical solution we need that the resulting solution $u(x, t)$ has continuous derivatives $u_x(x, t)$ and $u_t(x, t)$. This means that in particular $u_x(x, 0) = u'_0(x)$ must be continuous. Therefore we need to assume that the given initial function $u_0(x)$ is continuous and has a continuous derivative. If this is the case, and if the above condition on the existence interval of the ODE for $X(t)$ is satisfied one can then show that $u(x, t)$, $u_t(x, t)$ and $u_x(x, t)$ are continuous.

1.5 Method of Characteristics for PDE $u_t + c(x, t)u_x = f(x, t, u)$

Actually, the method of characteristics works in the same way for the more general case of the IVP

$$u_t + c(x, t)u_x = f(x, t, u), \quad u(x, 0) = u_0(x)$$

Note that the right hand side may contain nonlinear terms. Here is how we find **one** characteristic curve and the solution $u(x, t)$ on this characteristic curve:

Recipe: For a fixed value x_0 do the following:

1. Find the characteristic curve going through $(x_0, 0)$: Solve the IVP

$$X'(t) = c(X(t), t), \quad X(0) = x_0$$

2. Find the solution restricted to this characteristic curve: We can find the function $v(t) := u(X(t), t)$ by solving the IVP

$$v'(t) = f(X(t), t, v(t)), \quad v(0) = u_0(x_0).$$

Note that the ODE for $v(t)$ may now be nonlinear, so we need to assume that the solution exists for the times $t \in [0, T]$ we are interested in.

1.6 Method of Characteristics for quasilinear PDE $u_t + c(x, t, u)u_x = f(x, t, u)$

It turns out that we can generalize the method of characteristics to the case of so-called **quasilinear** 1st order PDEs:

$$u_t + c(x, t, u)u_x = f(x, t, u), \quad u(x, 0) = u_0(x) \quad (6)$$

Note that now both the left hand side and the right hand side may contain nonlinear terms. Assume that $u(x, t)$ is a solution of the initial value problem (6). If we already have this solution we can define characteristics by solving the IVP

$$X'(t) = c(X(t), t, u(X(t), t)), \quad X(0) = x_0.$$

We restrict the solution $u(x, t)$ to this characteristic curve and let $v(t) := u(X(t), t)$. Then we can write the IVP for $X(t)$ as

$$X'(t) = c(X(t), t, v(t)), \quad X(0) = x_0. \quad (7)$$

The function $v(t)$ satisfies

$$v'(t) = f(X(t), t, v(t)), \quad v(0) = u_0(x_0). \quad (8)$$

Note that we need $v(t)$ for (7), and we need $X(t)$ for (8), so the two ODEs are coupled and form a **system of 1st order ODEs**. But we now from the theory of ODEs that an initial value problem for a system of 1st order ODEs has a unique solution. It can also be solved numerically (e.g., using Euler method or Runge-Kutta method, or ode45 in Matlab). Here is how we find **one** characteristic curve and the solution $u(x, t)$ on this characteristic curve:

Recipe: For a fixed value x_0 do the following:

Find the functions $X(t), v(t)$ by solving the following IVP for a system of ODEs:

$$X'(t) = c(X(t), t, v(t)) \quad X(0) = x_0 \quad (9)$$

$$v'(t) = f(X(t), t, v(t)) \quad v(0) = u_0(x) \quad (10)$$

This system is nonlinear, so we need to assume that the solution exists for the times $t \in [0, T]$ we are interested in.

We can write the ODE system (9), (10) using vector notation: Let $\vec{w}(t) := \begin{bmatrix} X(t) \\ v(t) \end{bmatrix}$ so that $X(t) = w_1(t)$, $v(t) = w_2(t)$. Then the system can be written in the form

$$\vec{w}'(t) = \vec{G}(t, \vec{w}(t)), \quad \vec{w}(0) = \vec{w}^{(0)} \quad (11)$$

where we define

$$\vec{G}(t, \vec{w}) = \begin{bmatrix} c(w_1, t, w_2) \\ f(w_1, t, w_2) \end{bmatrix}, \quad \vec{w}^{(0)} = \begin{bmatrix} x_0 \\ u_0(x) \end{bmatrix}. \quad (12)$$

We will consider some examples where this problem is easy to solve. But in general it will not be possible to find a solution with pencil and paper, and we should use a numerical method.

1.7 Solving the ODE system for quasilinear PDE in Matlab

In Matlab we can use ode45 to solve an initial value problem numerically.

We first should define the functions $c(x, t, u)$, $f(x, t, u)$ and $u_0(x)$. We use the @-syntax to define functions in Matlab. E.g., for $c(t, x, u) = 4 - x$, $f(x, t, u) = u$ and $u_0(x) = e^{-x^2}$ we would use

```
c = @(x,t,u) 4-x
f = @(x,t,u) u
u0 = @(x) exp(-x^2)
```

We then can define the vector-valued function $\vec{G}(t, \vec{w})$ as in (12) by

```
G = @(t,w) [ c(w(1),t,w(2)) ; f(w(1),t,w(2)) ]
```

We pick a starting point x_0 and the times t_j for which I want to compute $X(t_j)$, $v(t_j)$. Let `tval` be the vector containing the values t_j . E.g.,

```
x0 = 0.7;
tval = 0:.1:3;
```

Now we can use `ode45` to solve the ODE system (11) by

```
[ts,ws] = ode45(G,tval,[x0;u0(x0)]);
```

In this case `ts` is the same as `tval`. The array `ws` has two columns, the first column contains the values $X(t_j)$, the second column contains the values of $v(t_j)$:

```
X=ws(:,1); v=ws(:,2);
```

We can now plot the characteristic consisting of the points $(X(t_j), t_j)$:

```
plot(X,tval,'.-')
```

The vector `v` now contains the solution values at those points, i.e., $v_j = u(X(t_j), t_j)$.

In order to find the solution at more points we just repeat this procedure for many starting points x_0 .

In the m-file **quasilin** (download from course web page) we specify a vector `xval` of initial points x_0 . We then have a loop iterating over those points. At the end we obtain an **array X** and an **array v** where each row corresponds to one value of x_0 ; each column corresponds to one value of t_j . Finally we plot all trajectories together, and we plot the graph of the solution u at time t_j by plotting the j -th column of `v` vs. the j -th column of `X`.

Usage of **quasilin**:

```
[X,v] = quasilin(c,f,u0,xval,tval)
```

Here `c`, `f`, `u0` are functions which should be defined using `@`:

```
c=@(x,t,u)...; f=@(x,t,u)...; u0=@(x)...
```

`xval` is the vector of x_0 -values, `tval` is the vector of t -values.

`X,v` are the arrays containing the coordinates of the characteristics and the values of v as explained above. We can plot the solution at time t_j by plotting column j of array `v` vs. column j of array `X`. E.g., to plot the solution at time `tval(3)` we use

```
plot(X(:,3),v(:,3))
```

Example for using **quasilin**:

Let us again consider the problem

$$u_t + (4-x)u_x = u, \quad u(x,0) = e^{-x^2}.$$

Here

$$c(t,x,u) = 4-x, \quad f(t,x,u) = u, \quad u_0(x) = e^{-x^2}.$$

Let us use for x_0 the x -values $-5, -4.9, \dots, 4.9, 5$ and the t -values $0, 0.1, \dots, 0.9, 1$:

```
c = @(x,t,u) 4-x;
f = @(x,t,u) u;
u0 = @(x) exp(-x^2);
[X,v] = quasilin(c,f,u0,-5:.1:5,0:.1:1);
```