1. Review: Eigenvalue problem for a symmetric matrix

1.1. Inner products

For vectors $u, v \in \mathbb{R}^n$ we define the *inner product* by

$$\langle u, v \rangle := \sum_{i=1}^{n} u_i v_i$$

and the norm by

$$\|u\| := \langle u, u \rangle .$$

We call two vectors u, v orthogonal if $\langle u, v \rangle = 0$.

Recall the Cauchy-Schwarz inequality: For any $u, v \in \mathbb{R}^n$

$$|\langle u, v \rangle| \leq \|u\| \, \|v\| . \tag{1}$$

For a matrix $A \in \mathbb{R}^{n \times n}$ we have

$$\langle Au, v \rangle = \left\langle u, A^\top v \right\rangle$$

where $A^\top$ denotes the transpose matrix. If $A = A^\top$ we call the matrix symmetric.

Lemma 1.1. *For a symmetric $A \in \mathbb{R}^{n \times n}$ we have*

$$\max_{\|u\|=1} |\langle Au, u \rangle| = \max_{\|u\|=1} \|Au\| \tag{2}$$

Proof. Let M_1 denote the value of the maximum on the left, let M_2 denote the value of the maximum on the right. By Cauchy-Schwarz (1) we have for any w

$$|\langle Aw, w \rangle| \leq \|Aw\| \, \|w\| \leq M_2 \, \|w\|^2 , \tag{3}$$

hence $M_1 \leq M_2$. Next we show $M_2 \leq M_1$: For an arbitrary u with $\|u\| = 1$ we let $\lambda := \|Au\|$, $f := \lambda u$, $g := Au$ and consider

$$X := \langle Af, g \rangle + \langle Ag, f \rangle = \lambda \, \langle Au, Au \rangle + \lambda \underbrace{\left\langle A^2 u, u \right\rangle}_{\langle Au, Au \rangle} = 2\lambda \, \|Au\|^2 = 2\lambda^3 .$$

On the other hand we see by multiplying out and using $\langle Aw, w \rangle \leq M_1 \, \|w\|^2$ that

$$X = \tfrac{1}{2} \langle A(f+g), f+g \rangle - \tfrac{1}{2} \langle A(f-g), f-g \rangle \leq \frac{M_1}{2} \langle f+g, f+g \rangle + \frac{M_1}{2} \langle f-g, f-g \rangle = M_1 \left(\|f\|^2 + \|g\|^2 \right) = M_1 2\lambda^2 ,$$

hence we have $X = 2\lambda^3 \leq M_1 2\lambda^2$. Therefore $\lambda = \|Au\| \leq M_1$ and hence $M_2 \leq M_1$ (since u with $\|u\| = 1$ is arbitrary). $\square$

Now consider an *invariant subspace* V of A. This means that $u \in V$ implies $Au \in V$. Then we obtain by the same argument

$$\max_{\substack{u \in V \\ \|u\|=1.}} |\langle Au, u \rangle| = \max_{\substack{u \in V \\ \|u\|=1.}} \|Au\| \tag{4}$$

1.2. Eigenvalue problem

Let A be a real $n \times n$ matrix. A number $\lambda \in \mathbb{C}$ is called an *eigenvalue* of the matrix if there exists a nonzero vector $v \in \mathbb{C}^n$ such that

$$Av = \lambda v. \tag{5}$$

All eigenvalues are real: Multiplying (5) from the left by $\overline{\vec{v}}^\top$ (bar denotes complex conjugate) gives $\overline{v}^\top Av = \lambda \overline{v}^\top v$. Since $\overline{A}^\top = A$ we have $\overline{v}^\top Av = \overline{(Av)}^\top v = \overline{\lambda}\overline{v}^\top v$ yielding $\lambda = \overline{\lambda}$.

Eigenvectors for different eigenvalues are orthogonal:

$$\lambda_j \left\langle v^{(j)}, v^{(k)} \right\rangle = \left\langle Av^{(j)}, v^{(k)} \right\rangle = \left\langle v^{(j)}, Av^{(k)} \right\rangle = \lambda_k \left\langle v^{(j)}, v^{(k)} \right\rangle, \text{ hence } \quad (\lambda_j - \lambda_k) \left\langle v^{(j)}, v^{(k)} \right\rangle = 0$$

The central result is that we have a full number of eigenvectors:

Theorem 1.2. *Assume that $A \in \mathbb{R}^{n\times n}$ is symmetric. Then*

1. *There are n real eigenvalues $\lambda_1, \ldots, \lambda_n$*
2. *There are corresponding real eigenvectors $v^{(1)}, \ldots, v^{(n)}$ which are orthogonal on each other.*
3. *Any vector $u \in \mathbb{R}^n$ can be written as a linear combination of eigenvectors:*

$$u = \sum_{j=1}^{n} c_j v^{(j)}, \qquad c_j = \frac{\left\langle u, v^{(j)} \right\rangle}{\left\langle v^{(j)}, v^{(j)} \right\rangle}$$

The key idea of the proof is to maximize $|\langle Au, u\rangle|$ over all u with $\|u\| = 1$. The solution of this problem yields an eigenvector:

Lemma 1.3. *Assume that $A \in \mathbb{R}^{n\times n}$ is symmetric and V is an invariant subspace, i.e., $v \in V$ implies $Av \in V$.*

1. *There exists $v \in V$ with $\|v\| = 1$ such that $|\langle Av, v\rangle|$ is maximal:*

$$|\langle Av, v\rangle| = M = \max_{\substack{u\in V \\ \|u\|=1.}} |\langle Au, u\rangle|$$

2. *Actually v is an eigenvector: There is $\lambda \in \mathbb{R}$ with*

$$Av = \lambda v, \qquad |\lambda| = M$$

Proof. Here we are maximizing a continuous function over a compact set, so there exists such a v with $\|v\| = 1$. Let $\lambda := \langle Av, v\rangle$. By (4) we have $\|Av\| \leq |\lambda|$ and hence

$$\|Av - \lambda v\|^2 = \|Av\|^2 - 2\lambda \underbrace{\langle Av, v\rangle}_{\lambda} + \lambda^2 \underbrace{\|v\|^2}_{1} \leq \lambda^2 - 2\lambda^2 + \lambda^2 = 0.$$

□

Now we can give the **proof of Theorem 1.2**:

First maximize $|\langle Au, u\rangle|$ over all $u \in \mathbb{R}^n$ with $\|u\| = 1$. By Lemma 1.3 this yields a vector $v^{(1)}$ and an eigenvalue λ_1. Define the space V_1 as all vectors which are orthogonal on $v^{(1)}$:

$$V_1 := \{u \in \mathbb{R}^n \mid \left\langle u, v^{(1)} \right\rangle = 0\}.$$

This is an invariant subspace: If $u \in V_1$, then Au satisfies

$$\left\langle Au, v^{(1)} \right\rangle = \left\langle u, Av^{(1)} \right\rangle = \lambda_1 \left\langle u, v^{(1)} \right\rangle \overset{u\in V_1}{=} 0,$$

hence $Au \in V_1$.

Now we maximize $|\langle Au, u\rangle|$ over all $u \in V_1$ with $\|u\| = 1$. By Lemma 1.3 this yields a vector $v^{(2)}$ and an eigenvalue λ_2. Define the space V_2 as all vectors which are orthogonal on $v^{(1)}$ and $v^{(2)}$:

$$V_2 := \{u \in \mathbb{R}^n \mid \left\langle u, v^{(j)} \right\rangle = 0 \text{ for } j = 1,2\}.$$

This is an invariant subspace: If $u \in V_2$, then Au satisfies for $j = 1,2$

$$\left\langle Au, v^{(j)} \right\rangle = \left\langle u, Av^{(j)} \right\rangle = \lambda_j \left\langle u, v^{(j)} \right\rangle \overset{u \in V_2}{=} 0,$$

hence $Au \in V_2$.

By continuing in the same way we obtain eigenvalues $\lambda_1, \dots, \lambda_n$ with

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$$

and eigenvectors $v^{(1)}, \dots, v^{(n)}$ which are by construction orthogonal on each other.

2. Review: Linear System of ODEs

2.1. Initial value problem

We consider a matrix of size $n \times n$ where the entries can depend on t: Let $A(t)$ be an $n \times n$ matrix with entries $a_{ij}(t)$ for $i, j = 1 \dots n$. We are given an initial vector $\vec{u}^{(0)}$ with entries $u_i^{(0)}$, $i = 1 \dots n$ and a right hand side function $\vec{f}(t)$ with entries $f_i(t)$, $i = 1 \dots n$. We want to find a function $\vec{u}(t)$ with entries $u_i(t)$, $i = 1 \dots n$ such that

$$\vec{u}'(t) + A(t)\vec{u}(t) = \vec{f}(t) \qquad \text{for } t \geq 0 \tag{6}$$

$$\vec{u}(0) = \vec{u}^{(0)} \tag{7}$$

If the function $\vec{f}(t)$ is zero we call the problem **homogeneous**.

If the functions $a_{ij}(t)$ and $f_i(t)$ are continuous for $t \geq 0$, then one can show that the initial value problem (6), (7) has a unique solution for all $t \geq 0$.

2.2. Homogeneous ODE with constant coefficients, symmetric matrix

We now consider a special case of (6), (7):

- $\vec{f}(t) = \vec{0}$ (homogeneous ODE)
- the matrix A is constant
- the matrix A is symmetric: $A = A^\top$.

Then we can solve the IVP in the following way:

1. **Find special solutions of the form** $u_i(t) = v_i g(t)$**:**

   We want to find solutions of the form $\vec{u}(t) = \vec{v}g(t)$ with a nonzero vector $\vec{v}$ and a function $g(t)$. Plugging this into the ODE gives

   $$\vec{v}g'(t) + A\vec{v}g(t) = \vec{0} \iff A\vec{v} = \frac{-g'(t)}{g(t)}\vec{v}$$

   (if we assume $g(t) \neq 0$). Since the left hand side does not depend on t and $\vec{v}$ is not the zero vector this equation can only hold if $-g'(t)/g(t)$ is equal to a constant λ.

Therefore we have to **solve an eigenvalue problem:** Find λ and a vector $\vec{v}$ (not the zero vector) such that

$$\boxed{A\vec{v} = \lambda\vec{v}}$$

Solving this eigenvalue problem gives **real eigenvalues** λ_j and **eigenvectors** $\vec{v}^{(j)}$ for $j = 1,\dots,n$ which are orthogonal on each other:

$$\left\langle \vec{v}^{(j)}, \vec{v}^{(k)} \right\rangle = 0 \quad \text{if } j \neq k. \tag{8}$$

For each eigenvector $\vec{v}^{(j)}$ we have now a special solution of the PDE: Since $-\frac{g'(t)}{g(t)} = \lambda_j$ we have $g(t) = c_j e^{-\lambda_j t}$ and our special solution is

$$c_j \vec{v} e^{-\lambda_j t}$$

for $j = 1,2,3,\dots$.

2. **Write the solution $\vec{u}(t)$ of the initial value problem as a linear combination of the special solutions:** We need to find coefficients $c_1,\dots,c_n$ such that

$$\boxed{\vec{u} = \sum_{j=1}^{\infty} c_j \vec{v}^{(j)} e^{-\lambda_j t}} \tag{9}$$

We find the coefficients by setting $t = 0$: Then the initial condition gives

$$\vec{u}^{(0)} = \sum_{j=1}^{\infty} c_j \vec{v}^{(j)} \tag{10}$$

We now take the inner product of this equation with the vector $\vec{v}^{(k)}$: Then the orthogonality (8) gives $\left\langle \vec{u}^{(0)}, \vec{v}^{(k)} \right\rangle = c_k \left\langle \vec{v}^{(k)}, \vec{v}^{(k)} \right\rangle$ or

$$\boxed{c_k = \frac{\left\langle \vec{u}^{(0)}, \vec{v}^{(k)} \right\rangle}{\left\langle \vec{v}^{(k)}, \vec{v}^{(k)} \right\rangle}} \tag{11}$$

2.3. Fundamental System $S(t)$

Let $\vec{u}^{(1)}(t)$ denote the solution of the homogeneous ODE with initial condition $\vec{u}^{(1)}(0) = [1,0,\dots,0]^\top, \dots$, Let $\vec{u}^{(n)}(t)$ denote the solution with initial condition $\vec{u}^{(n)}(0) = [1,0,\dots,0]^\top$. We define the $n \times n$ matrix

$$S(t) := \left[\vec{u}^{(1)}(t), \dots, \vec{u}^{(n)}(t) \right].$$

Now the solution of the homogeneous ODE with initial condition $\vec{u}(0) = \vec{u}^{(0)}$ is obtained as a linear combination of $\vec{u}^{(1)}(t),\dots,\vec{u}^{(n)}(t)$: We have $\vec{u}(t) = \vec{u}^{(1)}(t) \cdot u_1^{(0)} + \cdots + \vec{u}^{(n)}(t) \cdot u_n^{(0)}$, i.e.,

$$\vec{u}(t) = S(t)\vec{u}^{(0)}$$

2.4. Duhamel's principle

2.4.1. Review: Derivative of integral

We consider a function $F(t)$ defined as an integral

$$F(t) = \int_{a(t)}^{b(t)} f(s,t)\,ds.$$

Then the fundamental theorem of calculus and the chain rule imply that we have for the derivative

$$F'(t) = \int_{a(t)}^{b(t)} f_t(s,t)\,ds + f(b(t),t) \cdot b'(t) - f(a(t),t) \cdot a'(t). \tag{12}$$

2.4.2. General case

Duhamel's principle states: If we can solve the *homogeneous* initial value problem

$$\vec{u}'(t) + A(t)\vec{u}(t) = \vec{0} \qquad \text{for } t \geq s \tag{13}$$

$$\vec{u}(s) = \vec{u}^{(0)} \tag{14}$$

(here the initial condition is given at s instead of 0), then we can obtain a formula for the solution of the **inhomogeneous** initial value problem (6), (7) as follows:

1. Define the function $\vec{u}_{\text{hom}}(t)$ for $t \geq 0$ as the solution of the homogeneous initial value problem with initial condition

$$\vec{u}_{\text{hom}}(0) = \vec{u}^{(0)}$$

2. Define for any $s > 0$ the function $\vec{z}_{(s)}(t)$ for $t \geq s$ as the solution of the homogeneous initial value problem with initial condition

$$\vec{z}_{(s)}(s) = \vec{f}(s)$$

3. Now the solution $\vec{u}(x,t)$ of the inhomogeneous initial value problem (6), (7) is given by

$$\boxed{\vec{u}(t) = \vec{u}_{\text{hom}}(t) + \int_{s=0}^{t} \vec{z}_{(s)}(t)\,ds} \tag{15}$$

The proof is easy: For the particular solution $\vec{u}_{\text{part}}(t) := \int_{s=0}^{t} \vec{z}_{(s)}(t)ds$ we have obviously $\vec{u}_{\text{part}}(0) = \vec{0}$. Using (12) we obtain

$$\vec{u}'_{\text{part}}(t) = \int_{s=0}^{t} \vec{z}'_{(s)}(t)\,ds + \underbrace{\vec{z}_{(t)}(t)}_{\vec{f}(t)}$$

Since $\vec{z}'_{(s)}(t) + A\vec{z}_{(s)}(t) = \vec{0}$ we obtain $\vec{u}'_{\text{part}}(t) + A\vec{u}_{\text{part}}(t) = \vec{f}(t)$.

2.4.3. Case of ODE with constant coefficients

If we have a constant coefficient matrix A and a fundamental system $S(t)$ (which is an $n \times n$ matrix) we can use Duhamel's principle as follows:

1. The homogeneous solution is obtained as

$$\vec{u}_{\text{hom}}(t) = S(t)\vec{u}^{(0)}$$

2. The solution $\vec{z}_{(s)}(t)$ is obtained as

$$\vec{z}_{(s)}(t) = S(t-s)\vec{f}(s)$$

3. The solution $\vec{u}(x,t)$ of the inhomogeneous problem is therefore

$$\boxed{\vec{u}(t) = S(t)\vec{u}^{(0)} + \int_{s=0}^{t} S(t-s)\vec{f}(s)\,ds}$$

3. Heat Equation

3.1. Derivation as conservation law

We consider a long thin bar of some material. Let $u(x,t)$ denote the temperature at time t at position x along the bar. The temperature corresponds to heat energy stored in the bar. Let $r > 0$ denote the energy needed to increase the temperature of a bar of length 1 by 1 degree. This differs for different materials ("specific heat"). If the material (or the cross section of the bar) is not constant in x we have a function $r(x) > 0$. Then the total energy stored in the bar between $x = a$ and $x = b$ at time t is

$$E_{ab}(t) = \int_a^b r(x)u(x,t)dx \tag{16}$$

Because of the conservation of energy this can only change in time if heat flows across the endpoints a and b. Let $F(x,t)$ denote the heat flux at the point x at the time t, i.e., the amount of energy which flows across x (from left to right) per time unit. Therefore we must have

$$E'_{ab}(t) = F(a,t) - F(b,t) \tag{17}$$

since $F(a,t)$ is the energy per time coming in at the left endpoint, and $F(b,t)$ is the energy per time coming out at the right endpoint. We can now rewrite the left hand side and the right hand side of this equation as follows:

$$\int_a^b r(x)\partial_t u(x,t)dx = -\int_a^b \partial_x F(x,t)dx \iff \int_a^b \Big[r(x)\partial_t u(x,t) + \partial_x F(x,t) \Big] dx = 0.$$

This equation has to hold for all a, b along the bar, hence we must have

$$r(x)\partial_t u(x,t) + \partial_x F(x,t) = 0. \tag{18}$$

The heat flux $F(x,t)$ should be proportional to the change in temperature in x ("Fourier's law of cooling"). The energy should flow from the hotter to the colder region. Therefore we should have with a constant $\kappa > 0$ ("heat conductivity") that

$$F(x,t) = -\kappa \partial_x u(x,t). \tag{19}$$

If the material is not constant in x we may have a function $\kappa(x) > 0$ for the heat conductivity. Then we obtain from (18), (19) the PDE

$$r(x)\partial_t u(x,t) - \partial_x [\kappa(x)\partial_x u(x,t)] = 0.$$

This is the most general form of the heat equation. It is a linear homogeneous PDE. If we additionally have heat sources which give additional energy $f(x,t)$ per space and time unit at position x and time t we obtain the inhomogeneous PDE

$$\boxed{r(x)\partial_t u(x,t) - \partial_x [\kappa(x)\partial_x u(x,t)] = f(x,t)}$$

We will mainly consider the **case where the material properties are constant in x:** Then the functions r and κ are constant, and we have with $k := \kappa/r$ that

$$\boxed{u_t(x,t) - k u_{xx}(x,t) = f(x,t)}$$

3.1.1. Initial value problem for infinite bar

For an infinite bar we want to find a function $u(x,t)$ with $x \in \mathbb{R}$ and $t > 0$. We prescribe the initial temperature $u_0(x)$ at time zero. We obtain the following initial value problem:

Find a function $u(x,t)$ for $x \in \mathbb{R}$ and $t \geq 0$ such that

$$r(x)u_t(x,t) = \partial_x [\kappa(x)u_x(x,t)] \qquad \text{for } x \in \mathbb{R}, t > 0 \tag{20}$$

$$u(x,0) = u_0(x) \qquad \text{for } x \in \mathbb{R} \tag{21}$$

3.1.2. Initial value problem for bar of finite length

It is more realistic to consider a bar of finite length where $a \leq x \leq b$. In this case we want to find a function $u(x,t)$ with $x \in [a,b]$ and $t > 0$. We have to specify what happens at the boundary points $x = a$ and $x = b$. There are three reasonable choices for $x = a$:

- **Dirichlet condition (D):** We prescribe the temperature at $x = a$ for all times:

$$u(a,t) = g_1(t) \qquad \text{for } t > 0$$

  If the left end of the bar touches a large pool of ice water, we will have (approximately) $u(a,t) = 0$ (in degree Celsius). Note that in this case the heat flux across $x = a$ is in general nonzero, so the total energy in the bar is not conserved.

- **Neumann condition (N):** We prescribe the heat flux at $x = a$ for all times:

$$\kappa(a)u_x(a,t) = g_1(t) \qquad \text{for } t > 0$$

  If the left end of the bar touches a perfect insulator we will have $\kappa(a)u_x(a,t) = 0$. Note that in this case no energy crosses the boundary at $x = a$.

- **Robin condition (R):** Assume that we have a thin imperfect insulator with temperature $q(t)$ on the other side of the insulator. Then it is reasonable to assume that the heat flux to the left is proportional to the temperature difference on both sides of the insulator: With a constant $\alpha_1 > 0$ we have

$$\kappa(a)u_x(a,t) = -\alpha\,[u(a,t) - q(t)]$$

  or

$$\kappa(a)u_x(a,t) + \alpha_1 u(a,t) = g_1(t)$$

We also have three choices for the boundary condition at $x = b$:

- Dirichlet condition (D): $u(b,t) = g_2(t)$
- Neumann condition (N): $\kappa(b)u_x(b,t) = g_2(t)$
- Robin condition (R) with a constant $\alpha_2 > 0$: $\kappa(b)u_x(b,t) - \alpha_2 u(b,t) = g_2(t)$

We can have any of the 9 combinations (DD), (DN), (DR),... , (RR) for the boundary conditions at the left and right endpoints.

We obtain the following initial value problem:

Find a function $u(x,t)$ for $x \in \mathbb{R}$ and $t \geq 0$ such that for $t > 0$ we have the PDE

$$r(x)u_t(x,t) = \partial_x\,[\kappa(x)u_x(x,t)] \qquad \text{for } x \in (a,b)$$

and boundary conditions of the types (D), (N), (R) at $x = a$ and $x = b$. For $t = 0$ we have the initial condition

$$u(x,0) = u_0(x) \qquad \text{for } x \in (a,b).$$

We prescribe the initial temperature $u_0(x)$ at time zero.

Remark 3.1. If we have homogeneous Neumann conditions at both endpoints, i.e.,

$$u_x(a,t) = 0, \qquad u_x(b,t) = 0 \qquad \text{for } t > 0$$

then the heat flux at both endpoints is zero and (17) shows that the total energy $E_{ab}(t)$ must be constant:

$$\int_a^b r(x)u(x,t)dt = \int_a^b r(x)u_0(x)dx.$$

In all other cases we may have heat flow across one of the boundaries and therefore the total energy in $[a,b]$ may change.

3.2. Maximum Principle

We now consider the heat equation $u_t = ku_{xx}$ with a constant $k > 0$.

3.2.1. Maximum principle for interval

Theorem 3.2. *Assume $u(x,t)$ is a classical solution of the heat equation for $x \in [a,b]$, $t \in [0,T]$. This means that u is continuous for $x \in [a,b]$, $t \in [0,t]$, and the functions u_t, u_{xx} are continuous for $x \in (a,b)$, $t \in (0,T]$ and satisfy*

$$u_t(x,t) = ku_{xx}(x,t) \qquad \text{for } x \in (a,b), \quad t \in (0,T].$$

Then we have

$$\max_{\substack{x\in[a,b]\\ t\in[0,T]}} u(x,t) = \max\left\{\max_{x\in[a,b]} u(x,0), \max_{t\in[0,T]} u(a,t), \max_{t\in[0,T]} u(b,t)\right\}.$$

3.2.2. Maximum principle for real line

Theorem 3.3. *Assume $u(x,t)$ is a bounded classical solution of the heat equation for $x \in \mathbb{R}$, $t \in [0,T]$. This means that u is continuous for $x \in \mathbb{R}$, $t \in [0,T]$ and there exists C such that*

$$|u(x,t)| \le C \qquad x \in \mathbb{R}, \quad t \in [0,T].$$

Furthermore the functions u_t, u_{xx} are continuous for $x \in (a,b)$, $t \in (0,T]$ and satisfy

$$u_t(x,t) = ku_{xx}(x,t) \qquad \text{for } x \in (a,b), \quad t \in (0,T].$$

Then we have

$$\max_{\substack{x\in[a,b]\\ t\in[0,T]}} u(x,t) = \max_{x\in[a,b]} u(x,0).$$

3.2.3. Consequences

Note that we also have "minimum principles" with "min" instead of "max". (Just apply the maximum principle to $-u(x,t)$).

Uniqueness of solution for IVP on the real line: Assume we have two bounded classical solutions $u(x,t)$ and $\tilde{u}(x,t)$ which solve the same initial value problem on the real line. Then the difference $v(x,t) = u(x,t) - \tilde{u}(x,t)$ solves the initial value problem with zero initial data. By Theorem 3.3 we therefore must have $v(x,t) = 0$.

Uniqueness of solution for IBVP on an interval with Dirichlet conditions: Assume we have two classical solutions $u(x,t)$ and $\tilde{u}(x,t)$ which solve the same IBVP for an interval $x \in [a,b]$. Then the difference $v(x,t) = u(x,t) - \tilde{u}(x,t)$ solves the IBVP with zero initial data and zero boundary data. By Theorem 3.3 we therefore must have $v(x,t) = 0$.

Montonicity: Assume we have two initial functions $u_0(x)$ and $\tilde{u}_0(x)$ on the real line with

$$u_0(x) \le \tilde{u}_0(x) \qquad \text{for all } x \in \mathbb{R}.$$

Then we must have

$$u(x,t) \le \tilde{u}(x,t) \qquad \text{for all } x \in \mathbb{R}, t \ge 0$$

for the corresponding solutions of the IVP for the heat equation. (Just apply the maximum principle to $v(x,t) = \tilde{u}(x,t) - u(x,t)$.)

We obtain a similar monotonicity property for the IBVP on an interval with Dirichlet conditions.

"Well-posedness": This means that small changes in the given data only cause small changes in the solution. Consider the IBVP on an interval with Dirichlet conditions with given functions $u_0(x)$, $g_1(t)$, $g_2(t)$ which yields a solution $u(x,t)$. Consider slightly different given functions $\tilde{u}_0(x)$, $\tilde{g}_1(t)$, $\tilde{g}_2(t)$ with

$$|u_0(x)-\tilde{u}_0(x)| \leq \varepsilon, \qquad |g_1(t)-\tilde{g}_1(t)| \leq \varepsilon, \qquad |g_2(t)-\tilde{g}_2(t)| \leq \varepsilon.$$

We claim that we then must have for the corresponding solution $\tilde{u}(x,t)$ that

$$|u(x,t)-\tilde{u}(x,t)| \leq \varepsilon \qquad \text{for all } x \in [a,b],\, t \geq 0.$$

We can prove this by considering $u_{0,-}(x) := u_0(x) - \varepsilon$, $g_{1,-}(t) := g_1(t) - \varepsilon$, $g_{2,-}(t) := g_2(t) - \varepsilon$. By the monotonicity we must have for the corresponding solution $u_-(x,t)$ that

$$u(x,t) - \varepsilon \leq u_-(x,t) \leq \tilde{u}(x,t)$$

yielding $u(x,t) - \tilde{u}(x,t) \leq \varepsilon$. By considering $u_{0,+}(x) := u_0(x) + \varepsilon$ etc. we obtain $\tilde{u}(x,t) - u(x,t) \leq \varepsilon$.

3.3. Heat equation on the real line

3.3.1. Initial value problem for homogeneous case

We want to find a function $u(x,t)$ for $x \in \mathbb{R}$ and $t \geq 0$ such that

$$u_t(x,t) = k u_{xx}(x,t) \qquad \text{for } x \in \mathbb{R},\, t > 0 \tag{22}$$

$$u(x,0) = u_0(x) \qquad \text{for } x \in \mathbb{R} \tag{23}$$

For a **classical solution** the function $u(x,t)$ must be continuous for $x \in \mathbb{R}$, $t \geq 0$. Moreover, the derivatives u_t, u_{xx} must be continuous functions for $x \in \mathbb{R}$ and $t > 0$.

3.3.2. Invariance properties

Assume that $u(x,t)$ and $\tilde{u}(x,t)$ are solutions of the heat equation (22) for $x \in \mathbb{R}$, $t > 0$. Then we can obtain other solutions $v(x,t)$ of the heat equation as follows:

1. **Linearity of the problem:** For any $c_1, c_2 \in \mathbb{R}$

   $$v(x,t) := c_1 u(x,t) + c_2 \tilde{u}(x,t)$$

   is also a solution of the heat equation (22).

2. **Translation invariance in time:** For any $c \geq 0$

   $$v(x,t) := u(x,t+c)$$

   is also a solution of the heat equation (22).

3. **Translation invariance in space:** For any $c \in \mathbb{R}$

   $$v(x,t) := u(x-c,t)$$

   is also a solution of the heat equation (22).

4. **Dilation invariance:** For any $a > 0$

   $$v(x,t) := u(a^{1/2}x, at)$$

   is also a solution of the heat equation (22).

Note that the properties (1.) and (2.) also hold in the more general problem with functions $r(x)$, $\kappa(x)$, both for the problem on the infinite bar, and for the problem on the finite bar.

Note that the properties (3.), (4.) only hold in the case of constant r, κ and for an infinite bar.

3.3.3. Solution $U(x,t)$ where initial condition is step function

We will first consider the heat equation $u_t = u_{xx}$ with $k = 1$. We first consider the initial condition given by the Heaviside function (unit step function):

$$u_0(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases} \tag{24}$$

(we can define e.g. $u_0(0) = \frac{1}{2}$, though this does not really matter). We want to find a solution $U(x,t)$ of the heat equation (22) such that

$$\lim_{t\to 0} U(x,t) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases} \tag{25}$$

We assume that we have a solution $U(x,t)$ with this initial condition. We will analyze its properties which will lead us to a formula for $U(x,t)$.

Let

$$g(x) := U(x,1)$$

denote the solution at time $t = 1$. We want to show that the solution at any time $t > 0$ is a dilation of this function g. We use the dilation invariance of the heat equation: for any $a > 0$ the function

$$\tilde{U}(x,t) := U(a^{1/2}x, at)$$

also satisfies $\tilde{U}_t = \tilde{U}_{xx}$. Note that the function $\tilde{U}(x,t)$ satisfies the *same initial condition* (25) as $U(x,t)$. Therefore the difference $V(x,t) = U(x,t) - \tilde{U}(x,t)$ satisfies the heat equation with initial condition zero. Therefore we should have $V(x,t) = 0$.

[The function $V(x,t)$ is continuous for $t > 0$, but we don't know this for $x = 0$, $t = 0$. So we cannot directly apply our maximum principle. But one can use a slightly more general version to show $V(x,t) = 0$.]

Therefore we should have $U(x,t) = U(a^{1/2}x, at)$ for any $a > 0$. Choosing $a = t^{-1}$ gives

$$U(x,t) = U(t^{-1/2}x, 1) = g(t^{-1/2}x). \tag{26}$$

We can now find the function g by plugging $U(x,t) = g(t^{-1/2}x)$ into the PDE $U_t = U_{xx}$ and obtain

$$\begin{aligned} g'(t^{-1/2}x)(-\tfrac{1}{2})t^{-3/2}x &= g''(t^{-1/2}x)t^{-1} \\ -\tfrac{1}{2}(t^{-1/2}x)g'(t^{-1/2}x) &= g''(t^{-1/2}x) \\ -\tfrac{1}{2}sg'(s) &= g''(s) \end{aligned}$$

Let $\tilde{g}(t) := g'(t)$, then we obtain the first order linear ODE

$$\tilde{g}'(s) + \frac{s}{2}\tilde{g}(s) = 0$$

which we can easily solve by using the integrating factor $\mu(s) = e^{s^2/4}$ yielding the general solution

$$\begin{aligned} \tilde{g}(s) &= C_1 e^{-s^2/4} \\ g(s) &= C_1 \int_0^s e^{-p^2/4} dp + C_2 \end{aligned}$$

with $C_1, C_2 \in \mathbb{R}$. We now use the initial condition (25) for $U(x,t) = g(t^{-1/2}x)$: For $x > 0$ we must have $U(x,t) \to 1$ as $t \to 0$, i.e.,

$$g(s) \to 1 \qquad \text{as } s \to \infty.$$

For $x < 0$ we must have $U(x,t) \to 0$ as $t \to 0$, i.e.,

$$g(s) \to 0 \qquad \text{as } s \to -\infty.$$

Recall that

$$\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$$

and hence with the change of variables $p/2 = z$, $dp = 2dz$

$$\int_0^{\infty} e^{-p^2/4} dp = 2\int_0^{\infty} e^{-z^2} dz = \sqrt{\pi}. \tag{27}$$

Therefore we obtain

$$\lim_{s\to\infty} g(s) = C_1\sqrt{\pi} + C_2 \stackrel{!}{=} 1$$
$$\lim_{s\to-\infty} g(s) = -C_1\sqrt{\pi} + C_2 \stackrel{!}{=} 0$$

yielding $2C_2 = 1$ and $2C_1\sqrt{\pi} = 1$, i.e.,

$$g(s) = \tfrac{1}{2} + \tfrac{1}{2}\tfrac{1}{\pi}\int_0^s e^{-p^2/4} dp = \tfrac{1}{2} + \tfrac{1}{2}\tfrac{2}{\pi}\int_0^{s/2} e^{-z^2} dz = \tfrac{1}{2} + \tfrac{1}{2}\mathrm{erf}(s/2)$$

with the "error function" $\mathrm{erf}(x) := \frac{2}{\pi}\int_0^x e^{-x^2} dx$ (which is available in Matlab and most programming languages). Therefore we obtain for the solution of our initial value problem

$$\boxed{U(x,t) = g(t^{-1/2}x), \qquad \text{with } g(s) = \tfrac{1}{2} + \tfrac{1}{2}\mathrm{erf}(s/2)}$$

This function satisfies the heat equation $U_t(x,t) = U_{xx}(x,t)$ for $x \in \mathbb{R}$, $t > 0$. For $t \to 0$ it satisfies the initial condition

$$\lim_{t\to 0} U(x,t) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2} & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$

3.3.4. Fundamental solution

We define the **fundamental solution** $S(x,t)$ by taking the partial derivative with respect to x of $U(x,t)$:

$$S(x,t) := \partial_x U(x,t) = \partial_x g(t^{-1/2}x) = t^{-1/2} g'(t^{-1/2}x).$$

Recall that $g'(s) = \tilde{g}(s) = \dfrac{1}{2\sqrt{\pi}} e^{-s^2/4}$ yielding

$$\boxed{S(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right)}$$

The function $U(x,t)$ satisfies for $t > 0$ the heat equation $U_t(x,t) = U_{xx}(x,t)$. By taking the partial derivative of this with respect to x we obtain that $S_t(x,t) = S_{xx}(x,t)$, so $S(x,t)$ also satisfies the heat equation. The initial condition of $U(x,t)$ is the step function $u_0(x)$ in (24). Hence the initial condition of $S(x,t)$ should be the derivative $u_0'(x)$. The derivative of the step function does not exist as a classical function. However, in the sense of generalized functions (a.k.a. distributions) the derivative of the step function is the delta function: Formally, we have

$$S(x,0) = \delta(x).$$

The "delta function" has an infinitely sharp peak at $x = 0$, so that its integral is 1. Recall that the delta function formally satisfies for any continuous function f that

$$\int_{x=-\infty}^{\infty} f(x)\delta(x) dx = f(0).$$

Therefore we have for a continuous function $u_0(x)$

$$u_0(x) = \int_{y=-\infty}^{\infty} u_0(y)\delta(x-y)dy$$

i.e., we express u_0 formally as an "infinite linear combination of shifted delta functions". Analogously we now want to write the solution of the heat equation using the shifted fundamental solutions $S(x-y,t)$ multiplied by $u_0(y)$. We claim that the solution of the initial value problem with $u(x,0) = u_0(x)$ is given by

$$\boxed{u(x,t) = \int_{y=-\infty}^{\infty} S(x-y,t)u_0(y)dy}$$

We now have to prove that this formula really gives a solution for the initial value problem. We assume that the function $u_0(x)$ is continuous and bounded for $x \in \mathbb{R}$.

First we have to show that the integral exists as an improper integral for $t > 0$. This follows from the fact that $u_0(x)$ is bounded and the fast decay of $S(x,t)$ for $z \to \infty$ and $z \to -\infty$. Note that also the functions $\partial_t S(x,t)$ and $\partial_z S(z,t)$ decay fast for for $z \to \infty$ and $z \to -\infty$, hence also the integrals

$$\int_{y=-\infty}^{\infty} \partial_t S(x-y,t)u_0(y)dy, \qquad \int_{y=-\infty}^{\infty} \partial_x^2 S(x-y,t)u_0(y)dy$$

exist for $t > 0$ as improper integrals. One can see that these integrals must be equal to $u_t(x,t)$ and $u_{xx}(x,t)$, respectively. Since $S(x,t)$ satisfies the heat equation we have

$$\partial_t S(x-y,t) = \partial_x^2 S(x-y,t)$$

and hence $u_t(x,t) = u_{xx}(x,t)$.

Note that for any $t > 0$ with $z = t^{-1/2}x$, $dz = t^{-1/2}dx$

$$\int_{-\infty}^{\infty} S(x,t)dx = \frac{1}{2\sqrt{\pi t}} \int_{x=-\infty}^{\infty} \exp\left(-\frac{x^2}{4t}\right) dx = \frac{1}{2\sqrt{\pi}} \int_{z=-\infty}^{\infty} \exp\left(-\frac{z^2}{4}\right) dz = 1$$

by (27). This makes sense since the integral of the delta function is 1, i.e., initially we have total energy 1. Since the total energy on the real line is preserved we must have that the integral over the solution is 1 at all times. We also have

$$q(R) := \frac{1}{2\sqrt{\pi}} \int_{z=-R}^{R} \exp\left(-\frac{z^2}{4}\right) dz \to 1 \quad \text{as } R \to \infty. \tag{28}$$

$$\frac{1}{2\sqrt{\pi t}} \int_{x=-r}^{r} \exp\left(-\frac{x^2}{4t}\right) dx = q(t^{-1/2}r) \tag{29}$$

With the change of variables $w = x-y$, $dw = -dy$ we have

$$u(x,t) = \int_{y=-\infty}^{\infty} S(x-y,t)u_0(y)dy = \int_{w=-\infty}^{\infty} S(w,t)u_0(x-w)dw$$

Consider now a fixed point x_0. We want to show that $u(x_0,t) \to u_0(x_0)$ as $t \to 0$. We split the integral over $\mathbb{R}$ into two parts $(-t^{1/4}, t^{1/4})$ and $\mathbb{R} \setminus (-t^{1/4}, t^{1/4})$:

$$u(x,t) = I_1 + I_2$$

$$I_1 := \int_{-t^{1/4}}^{t^{1/4}} S(w,t)u_0(x_0-w)dw, \qquad I_2 := \int_{\mathbb{R}\setminus(-t^{1/4},t^{1/4})} S(w,t)u_0(x_0-w)dw$$

As $u_0(x)$ is bounded there exists C with $|u_0(x)| \le C$. We then have using (29)

$$|I_2| \le C \int_{\mathbb{R}\setminus(-t^{1/4},t^{1/4})} S(w,t)dw = C\left[1 - q(t^{-1/4})\right]$$

and we have for $t \to 0$

$$|I_2| \to C(1-1) = 0.$$

For I_1 we can find upper and lower bounds by using the maximum and minimum of $u_0(x)$:

$$\left(\min_{x_0 - t^{1/4} \le x \le x_0 + t^{1/4}} u_0(x)\right) q(t^{-1/4}) \le I_1 \le \left(\max_{x_0 - t^{1/4} \le x \le x_0 + t^{1/4}} u_0(x)\right) q(t^{-1/4})$$

For $t \to 0$ we have that the min and max must converge to $u_0(x_0)$ by the continuity of u_0, and we have that $q(t^{-1/4}) \to 1$, hence

$$I_1 \to u_0(x_0) \qquad \text{as } t \to 0.$$

Therefore we have for each fixed $x_0 \in \mathbb{R}$ that

$$\lim_{t \to 0} u(x,t) = u_0(x).$$

3.3.5. Problem with sources (inhomogeneous case)

For the inhomogeneous heat equation

$$u_t(x,t) - u_{xx}(x,t) = f(x,t) \qquad x \in \mathbb{R}, \quad t > 0$$

with heat sources $f(x,t)$ we can now apply the Duhamel principle:

1. We find the function $u_{\text{hom}}(x,t)$ by solving the homogeneous heat equation with initial condition at $t = 0$

$$u_{\text{hom}}(x,0) = u_0(x)$$

   yielding

$$u_{\text{hom}}(x,t) = \int_{y=-\infty}^{\infty} S(x-y,t) u_0(y) dy.$$

2. For $s > 0$ we find the function $z_{(s)}(x,t)$ by solving the homogeneous heat equation with initial condition at $t = s$

$$z_{(s)}(x,s) = f(x,s)$$

   yielding

$$z_{(s)}(x,t) = \int_{y=-\infty}^{\infty} S(x-y,t-s) f(y,s) dy.$$

3. Now the solution $u(x,t)$ of the inhomogeneous initial value problem (6), (7) is given by

$$u(x,t) = u_{\text{hom}}(x,t) + \int_{s=0}^{t} z_{(s)}(x,t) ds$$

$$\boxed{u(x,t) = \int_{y=-\infty}^{\infty} S(x-y,t) u_0(y) dy + \int_{s=0}^{t} \int_{y=-\infty}^{\infty} S(x-y,t-s) f(y,s) dy\, ds} \tag{30}$$

3.4. Heat equation on an interval

3.4.1. Formulation of the problem

We consider the heat equation for x in an interval $[a,b]$: That means $t > 0$ we want

$$u_t(x,t) = k u_{xx}(x,t) \qquad \text{for } x \in (a,b)$$

and boundary conditions at $x = a$ and $x = b$. In the **case (D)** of **Dirichlet conditions at both endpoints** this means that for $t > 0$

$$u(a,t) = 0, \qquad u(b,t) = 0.$$

In the **case (N)** of **Neumann conditions at both endpoints** this means that for $t > 0$

$$u_t(a,t) = 0, \qquad u_t(b,t) = 0.$$

We can also have **case (DN)** where we have a **Dirichlet condition at the left endpoint and a Neumann condition at the right endpoint**: For $t > 0$:

$$u(a,t) = 0, \qquad u_t(b,t) = 0.$$

We could also have a case (ND) where the conditions are the other way around. We can also have Robin boundary conditions $u'(a) + \alpha_1 u(a) = 0$ or $u'(b) - \alpha u(b) = 0$ with $\alpha_1, \alpha_2 > 0$.

In any of these cases we require that $u(x,t)$ satisfies an initial condition at $t = 0$:

$$u(0,t) = u_0(x) \qquad \text{for } x \in (a,b)$$

with a given function $u_0(x)$.

We say $u(x,t)$ is a **classical solution** if the function $u(x,t)$ is continuous for $x \in [a,b]$, $t \geq 0$, and the functions $u_t(x,t)$, $u_{xx}(x,t)$ are continuous for $x \in (a,b)$, $t > 0$.

3.5. Inner product and symmetry

We consider the heat equation on an interval $[a,b]$ with e.g. Dirichlet condition $u = 0$ for $x = a$ and Neumann condition $u_x = 0$ for $x = b$ (all other cases work in the same way).

We define the "**inner product**" of two functions u, v on the interval (a,b) as

$$\boxed{\langle u, v \rangle := \int_a^b u(x)v(x)dx}$$

Assume that the functions u, w have two continuous derivatives on $[a,b]$ and satisfy the boundary conditions

$$u(a) = 0 \qquad u'(b) = 0 \tag{31}$$
$$v(a) = 0 \qquad v'(b) = 0 \tag{32}$$

Let us write Au for $-u''$. We claim that the operator A with these boundary conditions is "symmetric" in the following sense:

$$\boxed{\langle Au, v \rangle = \langle u, Av \rangle}$$

This follows by using integration by parts twice:

$$-\int_a^b u''(x)v(x)dx = -\underbrace{\big[u'(x)v(x)\big]_a^b}_{0} + \int_a^b u'(x)v'(x)dx = \underbrace{\big[u(x)v'(x)\big]_a^b}_{0} - \int_a^b u(x)v''(x)dx$$

The terms at $x = a, b$ vanish because of the boundary conditions (31), (32).

3.6. Recipe for solution: "Separation of Variables"

We now consider the initial boundary value problem (IBVP) for the heat equation: Find a function $u(x,t)$ such that

$$\begin{aligned} u_t(x,t) - u_{xx}(x,t) = 0 \qquad x \in (a,b), \quad t > 0 \\ u(a,t) = 0, \qquad u'(b,t) = 0 \qquad t > 0 \end{aligned} \tag{33}$$

with the initial condition

$$u(x,0) = u_0(x) \qquad x \in (a,b)$$

for a given function $u_0(x)$. Note that with the operator $Au := -u_{xx}$ we can write the PDE formally as

$$\partial_t u + Au = 0$$

with boundary condition (33) and the initial condition

$$u(\cdot,0) = u_0.$$

This is reminiscent of the ODE initial value problem (6), (7). Therefore we want to use a similar strategy to solve the problem:

1. **Find special solutions of the form**

$$\boxed{u(x,t) = v(x)g(t)} \tag{34}$$

where $v(x)$ is not the zero function: Plugging this into the PDE gives

$$v(x)g'(t) = kv''(x)g(t) \tag{35}$$

where g satisfies the appropriate boundary conditions: E.g. in case (DN) we have

$$v(a) = 0, \qquad v'(b) = 0.$$

Assuming $g(t) \neq 0$ we can write (35) as

$$-kv''(x) = -\frac{g'(t)}{g(t)}v(x).$$

The left hand side does not depend on t. Since $v(x)$ is not the zero function this can only work if $-\dfrac{g'(t)}{g(t)}$ is a constant λ.

Therefore we have to **solve an eigenvalue problem:** Find λ and a function $v(x)$ (not the zero function) such that (in case (DN))

$$\boxed{\begin{aligned} -v''(x) = \lambda v(x) \qquad x \in (a,b) \\ v(a) = 0, \qquad v'(b) = 0 \end{aligned}} \tag{36}$$

Solving this eigenvalue problem gives **real eigenvalues** $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ with $\lambda_j \to \infty$ as $j \to \infty$, and **eigenfunctions** $v_j(x)$ which are orthogonal on each other:

$$\langle v_j, v_k \rangle = 0 \quad \text{if } j \neq k. \tag{37}$$

For each eigenfunction $v_j(x)$ we have now a special solution of the PDE: Since $-\dfrac{g'(t)}{g(t)} = \lambda_j$ we have $g(t) = c_j e^{-\lambda_j t}$ and our special solution is

$$c_j v_j(x) e^{-\lambda_j t}$$

for $j = 1,2,3,\ldots$.

2. **Write the solution $u(x,t)$ of the initial value problem as a linear combination of the special solutions:** We need to find coefficients $c_1, c_2, c_3, \dots$ such that

$$\boxed{u(x,t) = \sum_{j=1}^{\infty} c_j v_j(x) e^{-\lambda_j t}} \tag{38}$$

We find the coefficients by setting $t = 0$: Then the initial condition gives

$$u_0(x) = \sum_{j=1}^{\infty} c_j v_j(x) \tag{39}$$

We now take the inner product of this equation with the function $v_k(x)$: Then the orthogonality (37) gives $\langle u_0, v_k \rangle = c_k \langle v_k, v_k \rangle$ or

$$\boxed{c_k = \frac{\langle u_0, v_k \rangle}{\langle v_k, v_k \rangle}} \tag{40}$$

Summary of Recipe: We need to do the following steps:

- Find all eigenvalues λ_j and eigenfunctions $v_j(x)$ of the eigenvalue problem (36).
- Find the coefficients c_k by computing the integrals in (40).
- Then $u(x,t)$ is given by the series (42).

We claim that this recipe always gives the solution of our initial value problem.

Example 1: We want to solve the heat equation $u_t = u_{xx}$ for $x \in [0,1]$. We impose impose Dirichlet conditions at both endpoints (case (D))

$$u(0,t) = 0, \qquad u(1,t) = 0$$

and the initial condition

$$u(x,0) = u_0(x) = 1 \qquad x \in (0,1).$$

1. Solve the eigenvalue problem: Find $\lambda \in \mathbb{R}$ and function $v(x)$ (not zero function) such that

$$-v''(x) = \lambda v(x), \qquad v(0) = 0, \quad v(1) = 0.$$

For a given $\lambda \geq 0$ the ODE $v''(x) - \lambda v(x) = 0$ has the following general solution:

If $\lambda = 0$ the general solution is

$$v(x) = C_1 1 + C_2 x$$

Now $v(0) = 0$ gives $C_1 = 0$, and then $v(1) = 0$ gives $v(1) = 0$. Hence the only solution is $v(x) = 0$, and $\lambda = 0$ is not an eigenvalue.

If $\lambda > 0$ the general solution is

$$v(x) = C_1 \sin(\lambda^{1/2} x) + C_2 \cos(\lambda^{1/2} x).$$

Now $v(0) = 0$ gives $C_2 = 0$. Then $v(1) = 0$ gives $C_1 \sin(\lambda^{1/2}) = 0$. For a solution $v(x)$ which is not the zero function we need $\sin(\lambda^{1/2}) = 0$. We know that $\sin x = 0$ if and only if $x = j\pi$ for some integer j. Hence we need

$$\lambda^{1/2} = j\pi$$
$$\lambda = j^2\pi^2 \qquad j = 1, 2, 3, \dots$$

(recall that we want to find $\lambda > 0$). Therefore we have found the eigenvalues λ_j and eigenfunctions $v_j(x)$:

$$\lambda_j = j^2\pi^2, \qquad v_j(x) = \sin(j\pi x) \qquad j = 1, 2, 3, \dots$$

2. Find the coefficients c_k: We have to compute

$$c_k = \frac{\langle u_0, v_k \rangle}{\langle v_k, v_k \rangle} \qquad \text{for } k = 1, 2, 3, \ldots.$$

We first compute the denominator using $\int \sin^2 x\, dx = \frac{1}{2}x - \frac{1}{4}\sin(2x)$ and $z = k\pi x$:

$$\langle v_k, v_k \rangle = \int_0^1 \sin^2(k\pi x) dx = \frac{1}{k\pi}\left[\tfrac{1}{2}z - \tfrac{1}{4}\sin(2z)\right]_{z=0}^{k\pi} = \frac{1}{2}.$$

We then compute the numerator using $\cos(k\pi) = (-1)^k$ for integer k:

$$\langle u_0, v_k \rangle = \int_0^1 \sin(k\pi x) dx = \frac{1}{k\pi}\left[-\cos(k\pi x)\right]_{x=0}^{1} = \frac{1}{k\pi}\left(1 - (-1)^j\right)$$

Hence we obtain

$$c_k = \begin{cases} \frac{4}{k\pi} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even} \end{cases}$$

Therefore the solution of the initial value problem is

$$u(x,t) = \sum_{k=1,3,5,\ldots} \frac{4}{k\pi} \sin(k\pi x) e^{-k^2\pi^2 t}$$

Example 2: We want to solve the heat equation $u_t = u_{xx}$ for $x \in [0,1]$. We impose Neumann conditions at both endpoints (case (N))

$$u(0,t) = 0, \qquad u(1,t) = 0$$

and the initial condition

$$u(x,0) = u_0(x) = x \qquad x \in (0,1).$$

1. Solve the eigenvalue problem: Find $\lambda \in \mathbb{R}$ and function $v(x)$ (not zero function) such that

$$-v''(x) = \lambda v(x), \qquad v'(0) = 0, \quad v'(1) = 0.$$

For a given $\lambda \geq 0$ the ODE $v''(x) - \lambda v(x) = 0$ has the following general solution:

If $\lambda = 0$ the general solution is

$$v(x) = C_1 1 + C_2 x, \qquad v'(x) = C_2$$

Now $v'(0) = 0$ gives $C_2 = 0$, and then $v'(1) = 0$ is also satisfied. Hence we can obtain a solution (other than $v = 0$) as $v(x) = 1$. Therefore $\lambda_0 = 0$ is an eigenvalue, and the corresponding eigenfunction is $v_0(x) = 1$.

If $\lambda > 0$ the general solution is

$$v(x) = C_1 \sin(\lambda^{1/2}x) + C_2 \cos(\lambda^{1/2}x), \qquad v'(x) = C_1\lambda^{1/2}\cos(\lambda^{1/2}x) - C_2\lambda^{1/2}\sin(\lambda^{1/2}x)$$

Now $v'(0) = 0$ gives $C_1 = 0$. Then $v'(1) = 0$ gives $-C_2\lambda^{1/2}\sin(\lambda^{1/2}) = 0$. For a solution $v(x)$ which is not the zero function we need $\sin(\lambda^{1/2}) = 0$. We know that $\sin x = 0$ if and only if $x = j\pi$ for some integer j. Hence we need

$$\lambda^{1/2} = j\pi$$
$$\lambda = j^2\pi^2 \qquad j = 1, 2, 3, \ldots$$

(recall that we want to find $\lambda > 0$). Therefore we have found the eigenvalues λ_j and eigenfunctions $v_j(x)$:

$$\lambda_j = j^2\pi^2, \qquad v_j(x) = \sin(j\pi x) \qquad j = 1, 2, 3, \ldots$$

2. Find the coefficients c_k: We have to compute

$$c_k = \frac{\langle u_0, v_k\rangle}{\langle v_k, v_k\rangle} \qquad \text{for } k = 0,1,2,\ldots.$$

For $k = 0$ we obtain

$$c_0 = \frac{\langle u_0, v_0\rangle}{\langle v_0, v_0\rangle} = \frac{\int_0^1 x\,dx}{\int_0^1 1\,dx} = \frac{\frac{1}{2}}{1} = \tfrac{1}{2}.$$

For $k = 1,2,3,\ldots$ we have for the denominator using $\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x)$ and $z = k\pi x$:

$$\langle v_k, v_k\rangle = \int_0^1 \cos^2(k\pi x)dx = \frac{1}{k\pi}\left[\tfrac{1}{2}z + \tfrac{1}{4}\sin(2z)\right]_{z=0}^{k\pi} = \frac{1}{2}.$$

We then compute the numerator using the antiderivative $\int \cos x\,dx = \cos x + x\sin x$: Then the change of variables $z = k\pi x$ gives

$$\langle u_0, v_k\rangle = \int_{x=0}^1 x\cos(k\pi x)dx = \int_{z=0}^{k\pi} \frac{z}{k\pi}\cos(z)\frac{dz}{k\pi} = \frac{1}{k^2\pi^2}\left[\cos z + z\sin z\right]_{z=0}^{k\pi} = \frac{1}{k^2\pi^2}\left((-1)^k - 1\right) = \begin{cases} 0 & \text{for } k \text{ even} \\ \frac{-2}{k^2\pi^2} & \text{for } k \text{ odd} \end{cases}$$

Hence we obtain

$$c_k = \begin{cases} \frac{1}{2} & \text{for } k = 0 \\ \frac{-4}{k^2\pi^2} & \text{for } k \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Therefore the solution of the initial value problem is

$$u(x,t) = \tfrac{1}{2} + \sum_{k=1,3,5,\ldots} \frac{-4}{k^2\pi^2}\cos(k\pi x)e^{-k^2\pi^2 t}$$

3.7. Convergence of the series?

So far we have a formal recipe:

- Find all eigenvalues λ_j and eigenfunctions $v_j(x)$ of the eigenvalue problem (36).
- Find the coefficients c_k by computing the integrals in (40).
- Then $u(x,t)$ is given by the series (42).

Assume we are able to find all eigenvalues and eigenfunctions, and can compute all coefficients c_k. Then the following questions remain:

1. Does the series for $u_0(x)$

$$u_0(x) = \sum_{j=1}^{\infty} c_j v_j(x) \tag{41}$$

   converge? In which sense does it converge? Is the limit of the series really $u_0(x)$?

2. For $t > 0$: does the series

$$u(x,t) = \sum_{j=1}^{\infty} c_j v_j(x)e^{-\lambda_j t} \qquad x \in (a,b), \quad t > 0 \tag{42}$$

   converge? In which sense does it converge?

3. Does the function $u(x,t)$ defined by the series (42) satisfy the heat equation? This seams reasonable, since by construction each of the functions $v_j(x)e^{-\lambda_j t}$ are solutions of the heat equation. Therefore for a finite N the function $\sum_{j=1}^{N} c_j v_j(x)e^{-\lambda_j t}$ satisfies the heat equation. But for $N \to \infty$ one needs to prove this.

4. Does the function $u(x,t)$ satisfy the initial condition? I.e., if we let t tend to zero, will we have

$$\lim_{t\to 0} u(x,t) = u_0(x)$$

   For a finite series $\sum_{j=1}^{N}(\cdots)$ we can just let $t \to 0$ in each term. But for an infinite series this is more difficult.

3.8. Review: Uniform convergence

We consider a sequence $f_j(x),\ j = 1,2,3,\ldots$ of continuous functions for $x \in [a,b]$. We would like to know whether the limit function

$$f(x) = \lim_{j\to\infty} f_j(x) \tag{43}$$

exists and is also a continuous function.

Example: Let $f_j(x) = \tan^{-1}(jx)$ on the interval $[-1,1]$. Then for each $x \in [-1,1]$ the limit (43) exists:

$$\lim_{j\to\infty} \tan(jx) = \begin{cases} -\frac{\pi}{2} & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ \frac{\pi}{2} & \text{for } x > 0 \end{cases}$$

Obviously, the limit function $f(x)$ is not continuous. Here the functions $f_j(x)$ converge "pointwise", i.e., for each fixed x the sequence $f_j(x)$ is convergent.

A stronger notion of convergence of functions is *uniform convergence*: We say the functions $f_j(x)$ converge uniformly to the function $f(x)$ on the interval $[a,b]$ if

$$\max_{x\in[a,b]} \left|f_j(x) - f(x)\right| \to 0 \quad \text{as } j \to \infty$$

In this case the limit function must also be continuous:

Theorem 3.4. *Assume that the functions $f_j(x)$ are continuous for $x \in [a,b]$. Assume that for each $x \in [a,b]$ the limit*

$$f(x) := \lim_{j\to\infty} f_j(x)$$

exists, and that

$$\max_{x\in[a,b]} \left|f_j(x) - f(x)\right| \to 0 \quad \textit{as } j \to \infty$$

Then the function $f(x)$ is continuous.

Now we apply this theorem to the case of a series:

$$f(x) = \sum_{j=1}^{\infty} g_j(x) \tag{44}$$

Corollary 3.5. *Assume that the functions $g_j(x)$ are continuous for $x \in [a,b]$. Let*

$$\alpha_j := \max_{x\in[a,b]} \left|g_j(x)\right|$$

If

$$\sum_{j=1}^{\infty} \alpha_j < \infty$$

then $f_n = \sum_{j=1}^{n} g_j$ converges uniformly to f, and the limit function $f(x)$ is continuous on $[a,b]$.

Proof. For each $x \in [a,b]$ the limit (44) must exist since $\sum_{j=1}^{\infty} \left|g_j(x)\right|$ converges. We then have for any $x \in [a,b]$

$$|f(x) - f_n(x)| = \left|\sum_{j=n+1}^{\infty} g_j(x)\right| \le \sum_{j=n+1}^{\infty} \left|g_j(x)\right| \le \sum_{j=n+1}^{\infty} \alpha_j$$

Since $\sum_{j=1}^{\infty} \alpha_j$ converges the expression on the right hand side must converge to zero as $n \to \infty$. Therefore the sequence f_j satisfies the assumption of Theorem 3.4. □

3.9. Convergence for the examples

We want to see whether in the examples the series (39) for $u_0(x)$ and the series (38) for $u(x,t)$ ($t > 0$) converge uniformly.

Example 1: For $u_0(x) = 1$ we have obtained the series

$$u_0(x) = \sum_{k=1,3,5,\dots} \frac{4}{k\pi} \sin(k\pi x)$$

One can prove that the series converges to 1 for each fixed $x \in (0,1)$. For $x = 0$ we see that the series gives 0. Therefore the series cannot converge uniformly (if it did, it would converge to a continuous function.

Note that we cannot expect to find a classical solution of the heat equation for example 1: A classical solution must be continuous for $x \in [a,b]$ and $t \geq 0$. In this example we want $u(x,0) = 1$ for all x, and $u(0,t) = 0$ for all t. Therefore it is impossible for the function $u(x,t)$ to be continuous at $(x,t) = (0,0)$.

For $u(x,t)$ with $t > 0$ we have the series

$$u(x,t) = \sum_{k=1,3,5,\dots} \frac{4}{k\pi} e^{-k^2\pi^2 t} \sin(k\pi x)$$

Note that

$$\left| \frac{4}{k\pi} e^{-k^2\pi^2 t} \sin(k\pi x) \right| \leq \frac{4}{k\pi} e^{-k^2\pi^2 t} =: \alpha_k$$

Note that α_k decays exponentially fast if $t > 0$. Therefore

$$\sum_{k=1}^{\infty} \alpha_k < \infty$$

and by Corollary 3.5 the series for $u(x,t)$ converges uniformly. Now we consider $u_x(x,t)$: Taking the partial derivative with respect to x of each term of the series gives the series

$$\tilde{u}(x,t) := \sum_{k=1,3,5,\dots} \frac{4}{k\pi} e^{-k^2\pi^2 t} k \cos(k\pi x), \qquad \alpha_k = \frac{4}{k\pi} e^{-k^2\pi^2 t} k$$

Since $\sum_{k=1}^{\infty} \alpha_k < \infty$ the series converges uniformly, and we can also conclude that $u_x(x,t) = \tilde{u}(x,t)$. The same argument also works for u_{xx}, u_t and in fact for any partial derivative. In particular we obtain that

$$u_t(x,t) - u_{xx}(x,t) = \sum_{k=1,3,5,\dots} \frac{4}{k\pi} e^{-k^2\pi^2 t} \sin(k\pi x) \left[-k^2 + k^2 \right] = 0$$

Example 2: Here we obtained for $u_0(x) = x$ the series

$$u_0(x) = \tfrac{1}{2} + \sum_{k=1,3,5,\dots} \frac{-4}{k^2\pi^2} \cos(k\pi x)$$

Note that

$$\left| \frac{4}{k^2\pi^2} \cos(k\pi x) \right| \leq \frac{4}{k^2\pi^2} =: \beta_k$$

where $\sum_{k=1}^{\infty} \beta_k = \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$. Hence the series for $u_0(x)$ converges uniformly.

For $u(x,t)$ with $t > 0$ we have the series

$$u(x,t) = \tfrac{1}{2} + \sum_{k=1,3,5,\dots} \frac{-4}{k^2\pi^2} \cos(k\pi x) e^{-k^2\pi^2 t}.$$

Here

$$\left|\frac{-4}{k^2\pi^2}\cos(k\pi x)e^{-k^2\pi^2 t}\cos(k\pi x)\right| \le \frac{4}{k^2\pi^2}e^{-k^2\pi^2 t} =: \alpha_k$$

Note that α_k decays exponentially fast for $t > 0$. Therefore we obtain as in Example 1 that the series for $u(x,t)$ converges uniformly. The same holds for all partial derivatives. As in Example 1 it follows that $u(x,t)$ satisfies the heat equation.

But there is one difference: Note that α_k depends on t, but for any $t > 0$ we have the bound

$$\alpha_k = \frac{4}{k^2\pi^2}e^{-k^2\pi^2 t} \le \frac{4}{k^2\pi^2} = \beta_k$$

where $\sum_{k=1}^{\infty}\beta_k < \infty$. Therefore one is allowed to perform the limit for $t \to 0$ term by term:

$$\lim_{t\to 0} u(x,t) = \sum_{k=0}^{\infty}\lim_{t\to 0}\left[c_k v_k(x)e^{-\lambda_k t}\right] = \sum_{k=0}^{\infty} c_k v_k(x) = u_0(x).$$

Actually, one can see that this limit is uniform in x, so one obtains that the function $u(x,t)$ is actually continuous for $x \in [a,b]$ and $t \ge 0$. Therefore $u(x,t)$ is a classical solution of the heat equation.

A. Appendix: Hilbert space, orthogonal systems and eigenvalue problems

A.1. Hilbert space

We consider a real vector space H with the following properties

1. There is an **inner product** $\langle u,v\rangle$ satisfying the following: For any $u,v,w \in H$, $\alpha,\beta \in \mathbb{R}$

$$\langle \alpha u + \beta v, w\rangle = \alpha\langle u,v\rangle + \beta\langle v,w\rangle, \qquad \langle u,v\rangle = \langle v,u\rangle, \qquad \langle u,u\rangle > 0 \quad \text{for } u \ne 0$$

   We define the norm $\|u\| := \langle u,u\rangle^{1/2}$.

2. With respect to this norm the space H is **complete**: every Cauchy sequence converges.

Such a space is called a **Hilbert space**.

Recall that we call a sequence $u_1,u_2,u_3,\ldots$ of elements of H a **Cauchy sequence** if for any $\varepsilon > 0$ there exists N_0 such that

$$\forall M,N \ge N_0: \qquad \|u_M - u_N\| < \varepsilon.$$

Example 1: $H = \mathbb{R}^n$, $\langle u,v\rangle = \sum_{j=1}^{n} u_j v_j$

Example 2: H is the set of all sequences $(u_1,u_2,u_3,\ldots)$ with $\sum_{j=1}^{\infty} u_j^2 < \infty$. Here $\langle u,v\rangle = \sum_{j=1}^{\infty} u_j v_j$

Example 3: $H = L^2(I)$, the space of all functions u on an interval $I = (a,b)$ such that $\int_a^b u(x)^2 dx$ is bounded.

Remark: (i) One needs to use the Lebesgue integral for the proper definition. (ii) Functions u and v which differ only on a set of measure zero satisfy $\|u - v\| = 0$ and are considered identical.

A.2. Orthogonal vectors

We say vectors $v_1, v_2, v_3, \dots$ form an **orthogonal system** if $v_j \neq 0$ and

$$\left\langle v_j, v_k \right\rangle = 0 \quad \text{for } j \neq k$$

We can define the normalized vectors $w_j := v_j / \left\| v_j \right\|$. These vectors form an **orthonormal system**, i.e.,

$$\left\langle w_j, w_k \right\rangle = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

Assume that the vectors $w_1, \dots, w_N$ form an orthonormal system. We want to approximate a given vector $u \in H$ by a vector of the form

$$\tilde{u} = \sum_{j=1}^{N} c_j w_j$$

It is easy to see that the error $\|\tilde{u} - u\|$ becomes minimal if $\tilde{u} - u$ is orthogonal on all the vectors $w_1, \dots, w_n$, i.e.,

$$\left\langle \tilde{u} - u, w_j \right\rangle = 0 \qquad j = 1, \dots, N \tag{45}$$

As $\left\langle \tilde{u}, w_j \right\rangle = c_j$ this condition implies

$$c_j = \left\langle u, w_j \right\rangle.$$

Note that we have because of (45)

$$\|u\|^2 = \langle (u - \tilde{u}) + \tilde{u}, (u - \tilde{u}) + \tilde{u} \rangle = \|u - \tilde{u}\|^2 + \|\tilde{u}\|^2$$

As $\|\tilde{u}\|^2 = \sum_{j=1}^{N} c_j^2$ we have

$$\|u\|^2 = \|u - \tilde{u}\|^2 + \sum_{j=1}^{N} c_j^2. \tag{46}$$

Now we consider an infinite orthonormal system $w_1, w_2, w_3, \dots$. For a given vector $u \in H$ we define the coefficients $c_j := \left\langle u, w_j \right\rangle$ and

$$\tilde{u}_N := \sum_{j=1}^{N} c_j w_j.$$

By (46) we have $\sum_{j=1}^{N} c_j^2 \leq \|u\|^2$ and therefore we obtain for $N \to \infty$

$$\sum_{j=1}^{\infty} c_j^2 \leq \|u\|^2$$

This is called **Bessel inequality**. Since the series $S := \sum_{j=1}^{\infty} c_j^2$ converges, the sequence $S_N := \sum_{j=1}^{N} c_j^2$ must form a Cauchy sequence. Then the sequence $\tilde{u}_N$ must also form a Cauchy sequence since for $M \geq N$

$$\|\tilde{u}_M - \tilde{u}_N\|^2 = \sum_{j=N+1}^{M} c_j^2 = S_M - S_N.$$

As the space H is complete the sequence $\tilde{u}_N$ must have a limit $\tilde{u} \in H$. We then have

$$\|u\|^2 = \|u - \tilde{u}\|^2 + \|\tilde{u}\|^2 = \|u - \tilde{u}\|^2 + \sum_{j=1}^{\infty} c_j^2.$$

If we have for all $u \in H$ that

$$\sum_{j=1}^{\infty} c_j^2 = \|u\|^2$$

("**Parseval identity**") we call the system $w_1, w_2, w_3, \ldots$ a **complete orthonormal system**. In this case we have for all $u \in H$

$$\|u - \tilde{u}_N\| \to 0 \quad \text{as } N \to \infty.$$

Example 1: For the space $H = L^2[0,1]$ of square integrable functions on the interval $[0,1]$ the functions

$$w_j(x) = 2^{-1/2} \sin(\pi j x), \qquad j = 1, 2, 3, \ldots$$

form a complete orthonormal system.

Example 2: For the space $H = L^2[0,1]$ of square integrable functions on the interval $[0,1]$ the functions

$$w_j(x) = 2^{-1/2} \cos(\pi j x), \qquad j = 1, 2, 3, \ldots$$

form an orthonormal system, but it is not complete: E.g. for $u(x) = 1$ we have $\|u\| = 1$, but we have

$$c_k = \langle u, w_k \rangle = \int_0^1 \cos(\pi k x) dx = 0$$

and therefore $\tilde{u}(x) = 0$, $\sum_{j=1}^{\infty} c_j^2 = 0$. The Bessel inequality is satisfied, but we do not have the Parseval identity. We cannot represent all square integrable functions as a series $\tilde{u} = \sum_{j=1}^{\infty} c_j w_j$. Another way to see this is the following: All the functions $w_j(x)$ have integral zero: $\int_0^1 w_j(x) dx = 0$. Therefore we also must have $\int_0^1 \tilde{u}(x) dx = 0$.

Actually, one can show that one can obtain any function $u \in L^2[0,1]$ with integral zero as a series $u = \sum_{j=1}^{\infty} c_j w_j$. If we add to our orthonormal system the function 1 the new system

$$1, 2^{-1/2} \cos(\pi x), 2^{-1/2} \cos(2\pi x), 2^{-1/2} \cos(3\pi x), \ldots$$

is a complete orthonormal system for all of $L^2[0,1]$.

A.3. Linear operators

Let $T : H \to H$ be a linear mapping.

We call this mapping **bounded** if the set

$$S = \{Tu \mid u \in H, \|u\| \leq 1\}$$

is bounded, i.e., $C = \sup_{v \in S} \|v\|$ is finite.

We call a mapping **compact** if the set S is compact, i.e., every sequence in S has a convergent subsequence.

We call a linear mapping T **self-adjoint** if

$$\forall u, v \in H : \qquad \langle Tu, v \rangle = \langle u, Tv \rangle$$

A.4. The eigenvalue problem

We say $\mu \in \mathbb{R}$ is an eigenvalue of T if there exists a nonzero $v \in H$ such that

$$Tv = \mu v$$

In this case v is called the eigenvector for the eigenvalue μ.

Theorem A.1. *Assume that H is a Hilbert space and $T : H \to H$ is a linear compact self-adjoint operator. Assume that 0 is not an eigenvalue of T. Then*

1. *There are infinitely many eigenvalues $\mu_1, \mu_2, \mu_3, \ldots$. They are real and satisfy*

$$|\mu_1| \geq |\mu_2| \geq |\mu_3 \geq| \cdots, \qquad \mu_k \to 0 \quad \text{as } k \to \infty$$

2. *There are corresponding eigenvectors v_j which are orthogonal on each other:*

$$\langle v_j, v_k \rangle = 0 \quad \textit{for } j \neq k$$

3. *The eigenvectors form a complete orthogonal system: every $u \in H$ can be written as an infinite linear combination of eigenvectors: Let $c_j = \langle u, v_j \rangle / \langle v_j, v_j \rangle$, then*

$$u = \sum_{j=1}^{\infty} c_j v_j$$

This means that the $\left\| u - \sum_{j=1}^{N} c_j v_j \right\| \to 0$ as $N \to \infty$.

I don't give the proof of this theorem. But it is similar to the proof of Theorem 1.2: We first maximize $|\langle Tu, u \rangle|$ over all $u \in H$ with $\|u\| = 1$ yielding a vector v_1. One can again show that this function must be an eigenvector with eigenvalue μ_1. Then one considers the subspace $V_1 = \{u \in H \mid \langle u, v_1 \rangle = 0\}$ of all vectors orthogonal to v_1. Now we maximize $|\langle Tu, u \rangle|$ over all $u \in V_1$ with $\|u\| = 1$ yielding a vector v_2. One obtains that this function must be an eigenvector with eigenvalue μ_2 where $|\mu_2| \leq |\mu_1|$. In this way one obtains all eigenvectors $v_1, v_2, v_3, \ldots$.

A.5. Application to the ODE eigenvalue problem

Let us consider e.g. the Dirichlet case: Find λ and a function v such that

$$\begin{aligned} -v''(x) &= \lambda v(x) \qquad x \in (a,b) \\ v(a) = 0,& \quad v(b) = 0 \end{aligned} \tag{47}$$

We first consider the BVP with a given right hand side function $f(x)$: Find $u(x)$ such that

$$\begin{aligned} -u''(x) &= f(x) \qquad x \in (a,b) \\ u(a) = 0,& \quad u(b) = 0 \end{aligned}$$

In this case it is easy to see that we can obtain the unique solution by taking the antiderivative twice and adjust the constants to satisfy the boundary conditions. We denote the operator which maps the function f to the function u by T: We have

$$u = Tf$$

If u satisfies $-u'' = f$ and the boundary conditions, w satisfies $-w'' = g$ and the boundary conditions we have

$$\begin{aligned} \langle -u'', w \rangle &= \langle u, -w'' \rangle \\ \langle f, Tg \rangle &= \langle Tf, g \rangle \end{aligned}$$

Therefore the operator T is self-adjoint. One can also show that the operator can be defined for any $f \in L^2$ and gives a function $u = Tf \in L^2$. Furthermore one can show that the operator T is compact. Therefore Theorem A.1 shows that the operator T has infinitely many eigenvalues μ_j with eigenfunctions v_j such that

$$Tv_j = \mu_j v_j$$

Hence $\mu_j v_j$ satisfies $-\mu_j v_j'' = v_j$. Therefore v_j is an eigenfunction of our BVP (47) with $\lambda_j := \mu_j^{-1}$:

$$-v_j'' = \lambda_j v_j$$

The eigenvalues μ_j of the operator T satisfy $\lim_{j\to\infty} \mu_j = 0$. Hence we obtain that the eigenvalues λ_j satisfy $\lim_{j\to\infty} \lambda_j = \infty$.

A.6. The general case with coefficient functions $r(x)$, $\kappa(x)$

Recall from 3.1 that for the case of variable material properties along the bar we obtain the heat equation in the form

$$r(x)u_t(x,t) - \partial_x[\kappa(x)u_x(x,t)] = 0 \qquad x \in (a,b) \tag{48}$$

with functions $r(x)$ (specific heat) and $\kappa(x)$ (heat conductivity) where $r(x) > 0$, $\kappa(x) > 0$ for $x \in [a,b]$. Let us consider the case of the Dirichlet problem (the other cases work in the same way) with the boundary conditions

$$u(a,t) = 0, \qquad u(b,t) = 0 \qquad t > 0. \tag{49}$$

We want to find a solution $u(x,t)$ which satisfies an initial condition

$$u(x,0) = u_0(x) \qquad x \in (a,b). \tag{50}$$

Then the separation of variables method leads to the following eigenvalue problem:

Find λ and a function $v(x)$ (not the zero function) such that

$$\begin{aligned} -\frac{d}{dx}\left[\kappa(x)v'(x)\right] &= \lambda r(x)v(x) \qquad x \in (a,b) \\ v(a) = 0, &\qquad v(b) = 0 \end{aligned} \tag{51}$$

We now define the inner product $\langle u, v\rangle$ in a different way:

$$\langle u, v\rangle := \int_a^b u(x)v(x)r(x)dx.$$

If A denotes the operator

$$Au := -r(x)^{-1}\frac{d}{dx}\left[\kappa(x)v'(x)\right]$$

we claim that the operator is self-adjoint for functions u, v which are zero at $x = a,b$:

$$\langle Au, v\rangle = -\int_a^b \frac{d}{dx}\left[\kappa(x)u'(x)\right]v(x)dx = -\underbrace{\left[\kappa(x)u'(x)v(x)\right]_a^b}_{0} + \int_a^b u'(x)\kappa(x)v'(x)dx$$

$$= \underbrace{\left[\kappa(x)u(x)v'(x)\right]_a^b}_{0} - \int_a^b u(x)\frac{d}{dx}\left[\kappa(x)v'(x)\right]dx = \langle u, Av\rangle$$

Therefore we can use the standard recipe to solve the IBVP (48), (49), (50):

- Find all eigenvalues λ_j and eigenfunctions $v_j(x)$ of the eigenvalue problem (51).
- Find the coefficients c_k using

$$c_k = \frac{\langle u_0, v_k\rangle}{\langle v_k, v_k\rangle} = \frac{\int_a^b u_0(x)v_k(x)r(x)dx}{\int_a^b v_k(x)^2 r(x)dx}$$

- Then $u(x,t)$ is given by the series

$$u(x,t) = \sum_{k=1}^{\infty} c_k v_k(x)e^{-\lambda_k t}$$

Remark A.2. The same method also works for the PDE with an additional term $q(x)u(x,t)$

$$r(x)u_t(x,t) - \partial_x[\kappa(x)u_x(x,t)] + q(x)u(x,t) = 0 \qquad x \in (a,b)$$

where the function q satisfies $q(x) \geq 0$. In this case we have to solve the eigenvalue problem

$$\begin{aligned} -\frac{d}{dx}\left[\kappa(x)v'(x)\right] + q(x)u(x) &= \lambda r(x)v(x) \qquad x \in (a,b) \\ v(a) = 0, &\qquad v(b) = 0. \end{aligned}$$