

Solving the heat equation, wave equation, Poisson equation using separation of variables and eigenfunctions

1 Review: Interval in one space dimension

Our domain $G = (0, L)$ is an **interval** of length L . The boundary $\partial G = \{0, L\}$ are the two endpoints. We consider here as an example the case (DD) of **Dirichlet boundary conditions**: Dirichlet conditions at $x = 0$ and $x = L$. For other boundary conditions (NN), (DN), (ND) one can proceed similarly.

In one dimension the Laplace operator is just the second derivative with respect to x : $\Delta u(x, t) = u_{xx}(x, t)$. We will consider three different problems:

- heat equation $u_t - \Delta u = f$ with boundary conditions, initial condition for u
- wave equation $u_{tt} - \Delta u = f$ with boundary conditions, initial conditions for u, u_t
- Poisson equation $-\Delta u = f$ with boundary conditions

Here we use constants $k = 1$ and $c = 1$ in the wave equation and heat equation for simplicity. But the case with general constants k, c works in the same way.

1.1 Heat equation on an interval

We want to find a function $u(x, t)$ for $x \in G$ and $t \geq 0$ such that

$$\begin{aligned}u_t(x, t) - u_{xx}(x, t) &= f(x, t) & x \in G, \quad t > 0 \\u(x, t) &= 0 & x \in \partial G, \quad t > 0 \\u(x, 0) &= u_0(x), & x \in G\end{aligned}$$

with a given functions $f(x, t)$ and $u_0(x)$. This describes e.g. heat conduction in a metal bar where $u(x)$ is the temperature.

1.2 Wave equation on an interval

We want to find a function $u(x, t)$ for $x \in G$ and $t \geq 0$ such that

$$\begin{aligned}u_{tt}(x, t) - u_{xx}(x, t) &= f(x, t) & x \in G, \quad t > 0 \\u(x, t) &= 0 & x \in \partial G, \quad t > 0 \\u(x, 0) &= u_0(x), & x \in G \\u_t(x, 0) &= u_1(x), & x \in G\end{aligned}$$

with a given functions $f(x, t), u_0(x), u_1(x)$. This describes e.g. an elastic string where $u(x)$ is the displacement.

1.3 Poisson equation on an interval

Now we consider a given function $f(x)$ which only depends on x . We want to find a function $u(x)$ for $x \in G$ such that

$$\begin{aligned}-u_{xx}(x) &= f(x) & x \in G \\u(x) &= 0 & x \in \partial G\end{aligned}$$

This describes the equilibrium problem for either the heat equation or the wave equation, i.e., temperature in a bar at equilibrium, or displacement of a string at equilibrium.

1.4 Eigenvalue problem for Laplace operator on an interval

For all three problems (heat equation, wave equation, Poisson equation) we first have to solve an **eigenvalue problem**: Find functions $v(x)$ and numbers λ such that

$$\begin{aligned} -v''(x) &= \lambda v(x) & x \in G \\ v(x) &= 0, & x \in \partial G \end{aligned}$$

We will always have $\lambda \geq 0$. In the case (NN) of pure Neumann conditions there is an eigenvalue $\lambda = 0$, in all other cases (as in the case (DD) here) we have $\lambda > 0$.

For $\lambda > 0$ the general solution of the ODE $-v''(x) - \lambda v(x) = 0$ is given by

$$v(x) = C_1 \cos(\lambda^{1/2}x) + C_2 \sin(\lambda^{1/2}x)$$

The boundary condition at $v(0) = 0$ implies $C_1 = 0$. Then the boundary condition $v(L) = 0$ gives $\lambda^{1/2}L = j\pi$ for $j = 1, 2, 3, \dots$. Therefore we have found the eigenvalues λ_j and eigenfunctions $v_j(x)$

$$\lambda_j = \left(\frac{j\pi}{L}\right)^2, \quad v_j(x) = \sin\left(\frac{j\pi}{L}x\right) \quad j = 1, 2, 3, \dots$$

The eigenfunctions have the following properties:

- **orthogonality**: With the **inner product** $\langle f, g \rangle := \int_G f(x)g(x)dx$ we have

$$\langle v_j, v_k \rangle = 0 \quad \text{for } j \neq k$$

- **completeness**: For any $F(x)$ where $\int_G F(x)^2 dx$ exists we can define the **Fourier coefficients**

$$F_j := \frac{\langle F, v_j \rangle}{\langle v_j, v_j \rangle}$$

and then have that the **Fourier series** converges to $F(x)$

$$F(x) = \sum_{j=1}^{\infty} F_j v_j(x)$$

in the sense that $\left\| F - \sum_{j=1}^N F_j v_j \right\| \rightarrow 0$ as $N \rightarrow \infty$. Here $\|g\| := \langle g, g \rangle^{1/2}$.

Note that for the function $v_j(x) = \sin\left(\frac{j\pi}{L}x\right)$ we have

$$\langle v_j, v_j \rangle = \int_{x=0}^L \sin^2\left(\frac{j\pi}{L}x\right) dx = \frac{L}{j\pi} \int_{z=0}^{j\pi} \sin^2 z dz = \frac{L}{2} \quad (1)$$

1.5 Solution of the Poisson problem on an interval

We can represent the solution $u(x)$ in terms of its Fourier series

$$u(x) = \sum_{j=1}^{\infty} u_j v_j(x)$$

where we need to find the coefficients u_j . Therefore we plug this into the ODE $-u''(x) = f(x)$

$$-u''(x) = \sum_{j=1}^{\infty} u_j (-v_j''(x)) = \sum_{j=1}^{\infty} u_j \lambda_j v_j(x) \stackrel{!}{=} f(x)$$

and obtain that $u_j \lambda_j$ are the Fourier coefficients of the function $f(x)$, i.e.,

$$u_j := \lambda_j^{-1} \frac{\langle f, v_j \rangle}{\langle v_j, v_j \rangle}$$

1.6 Solution of the heat equation on an interval

We first consider the heat equation $u_t - u_{xx} = 0$ with $f(x, t) = 0$. The separation of variables method means that we first look for special solutions of the form $u(x, t) = g(t)v(x)$ and obtain

$$u_t - u_{xx} = g'(t)v(x) - g(t)v''(x) = 0$$

$$-\frac{g'(t)}{g(t)} = -\frac{v''(x)}{v(x)} = \lambda$$

$$-v''(x) = \lambda v(x), \quad v(0) = 0, \quad v(L) = 0 \quad (2)$$

$$g'(t) + \lambda g(t) = 0 \quad (3)$$

Equation (2) means that $v(x)$ is an eigenfunction $v_j(x)$ and λ is an eigenvalue λ_j , $j = 1, 2, \dots$. Equation (3) is an ODE with general solution

$$g(t) = C_j e^{-\lambda_j t}.$$

Therefore the special solutions with separated variables are

$$C_j e^{-\lambda_j t} v_j(x), \quad j = 1, 2, 3, \dots$$

For the solution of our initial value problem $u_t - u_{xx} = 0$ with initial condition $u(x, 0) = u_0(x)$ we want to write the solution as a linear combination of these special solutions:

$$u(x, t) = \sum_{j=1}^{\infty} C_j e^{-\lambda_j t} v_j(x)$$

This will satisfy the PDE, but we also need to satisfy the initial condition:

$$u(x, 0) = \sum_{j=1}^{\infty} C_j v_j(x) \stackrel{!}{=} u_0(x)$$

Therefore C_j must be the Fourier coefficients of the function $u_0(x)$:

$$C_j := \frac{\langle u_0, v_j \rangle}{\langle v_j, v_j \rangle} \quad (4)$$

For the general problem $u_t - u_{xx} = f(x, t)$ with a source function $f(x, t)$ we can then use Duhamel's principle:

$$u(x, t) = u_{\text{hom}}(x, t) + u_{\text{part}}(x, t)$$

$$u_{\text{hom}}(x, t) = \sum_{j=1}^{\infty} C_j e^{-\lambda_j t} v_j(x), \quad C_j \text{ given by (4)}$$

$$u_{\text{part}}(x, t) = \int_{s=0}^t z_{(s)}(x, t) ds$$

$$z_{(s)}(x, t) = \sum_{j=1}^{\infty} F_j(s) e^{-\lambda_j(t-s)} v_j(x), \quad F_j(s) := \frac{\langle f(\cdot, s), v_j \rangle}{\langle v_j, v_j \rangle}$$

1.7 Solution of the wave equation on an interval

We first consider the wave equation $u_{tt} - u_{xx} = 0$ with $f(x, t) = 0$. The separation of variables method means that we first look for special solutions of the form $u(x, t) = g(t)v(x)$ and obtain

$$u_{tt} - u_{xx} = g''(t)v(x) - g(t)v''(x) = 0$$

$$-\frac{g''(t)}{g(t)} = -\frac{v''(x)}{v(x)} = \lambda$$

$$-v''(x) = \lambda v(x), \quad v(0) = 0, \quad v(L) = 0 \quad (5)$$

$$g''(t) + \lambda g(t) = 0 \quad (6)$$

Equation (5) means that $v(x)$ is an eigenfunction $v_j(x)$ and λ is an eigenvalue λ_j , $j = 1, 2, \dots$. Equation (6) is for $\lambda_j > 0$ an ODE with general solution

$$g(t) = A_j \cos(\lambda_j^{1/2} t) + B_j \sin(\lambda_j^{1/2} t)$$

Therefore the special solutions with separated variables are

$$\left[A_j \cos(\lambda_j^{1/2} t) + B_j \sin(\lambda_j^{1/2} t) \right] v_j(x), \quad j = 1, 2, 3, \dots$$

For the solution of our initial value problem $u_t - u_{xx} = 0$ with initial conditions $u(x, 0) = u_0(x)$ and $u_t(x, 0) = u_1(x)$ we want to write the solution as a linear combination of these special solutions:

$$u(x, t) = \sum_{j=1}^{\infty} \left[A_j \cos(\lambda_j^{1/2} t) + B_j \sin(\lambda_j^{1/2} t) \right] v_j(x)$$

This will satisfy the PDE, but we also need to satisfy the initial conditions:

$$\begin{aligned} u(x, 0) &= \sum_{j=1}^{\infty} A_j v_j(x) \stackrel{!}{=} u_0(x) \\ u_t(x, 0) &= \sum_{j=1}^{\infty} B_j \lambda_j^{1/2} v_j(x) \stackrel{!}{=} u_1(x) \end{aligned}$$

Therefore A_j must be the Fourier coefficients of the function $u_0(x)$, and $\lambda_j^{1/2} B_j$ must be the Fourier coefficients of the function $u_1(x)$:

$$A_j := \frac{\langle u_0, v_j \rangle}{\langle v_j, v_j \rangle}, \quad B_j := \lambda_j^{-1/2} \frac{\langle u_1, v_j \rangle}{\langle v_j, v_j \rangle} \quad (7)$$

For the general problem $u_{tt} - u_{xx} = f(x, t)$ with a source function $f(x, t)$ we can then use Duhamel's principle:

$$\begin{aligned} u(x, t) &= u_{\text{hom}}(x, t) + u_{\text{part}}(x, t) \\ u_{\text{hom}}(x, t) &= \sum_{j=1}^{\infty} \left[A_j \cos(\lambda_j^{1/2} t) + B_j \sin(\lambda_j^{1/2} t) \right] v_j(x), \quad A_j, B_j \text{ given by (7)} \\ u_{\text{part}}(x, t) &= \int_{s=0}^t z_{(s)}(x, t) ds \\ z_{(s)}(x, t) &= \sum_{j=1}^{\infty} F_j(s) \sin(\lambda_j^{1/2} (t-s)) v_j(x), \quad F_j(s) := \lambda_j^{-1/2} \frac{\langle f(\cdot, s), v_j \rangle}{\langle v_j, v_j \rangle} \end{aligned}$$

2 Rectangle in two space dimensions

We now consider the case of two space dimensions where the domain G is a **rectangle** $G = (0, L_1) \times (0, L_2)$. The boundary ∂G consists of the four sides of the rectangle. We consider here the case of **Dirichlet boundary conditions** on all four sides of the rectangle. We could also consider any combination of Dirichlet and Neumann conditions for the four sides (e.g., Dirichlet conditions on one side and Neumann conditions on the other three sides) and proceed similarly.

In two dimensions the Laplace operator is $\Delta u(x, y, t) = u_{xx}(x, y, t) + u_{yy}(x, y, t)$. We will again consider three different problems:

- heat equation $u_t - \Delta u = f$ with boundary conditions, initial condition for u
- wave equation $u_{tt} - \Delta u = f$ with boundary conditions, initial conditions for u, u_t
- Poisson equation $-\Delta u = f$ with boundary conditions

Note that the analogous problems in three dimensions on a box $G = (0, L_1) \times (0, L_2) \times (0, L_3)$ can be solved with the same method.

2.1 Heat equation on a rectangle

We want to find a function $u(x, y, t)$ for $(x, y) \in G$ and $t \geq 0$ such that

$$u_t(x, y, t) - \Delta u(x, y, t) = f(x, t) \quad (x, y) \in G, \quad t > 0 \quad (8)$$

$$u(x, y, t) = 0 \quad (x, y) \in \partial G, \quad t > 0 \quad (9)$$

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in G \quad (10)$$

with a given functions $f(x, t)$ and $u_0(x)$. This describes e.g. heat conduction in a rectangular metal plate where $u(x)$ is the temperature.

2.2 Wave equation on a rectangle

We want to find a function $u(x, y, t)$ for $(x, y) \in G$ and $t \geq 0$ such that

$$u_t(x, y, t) - \Delta u(x, y, t) = f(x, t) \quad (x, y) \in G, \quad t > 0 \quad (11)$$

$$u(x, y, t) = 0 \quad (x, y) \in \partial G, \quad t > 0 \quad (12)$$

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in G \quad (13)$$

$$u_t(x, y, 0) = u_1(x, y), \quad (x, y) \in G \quad (14)$$

with a given functions $f(x, t)$, $u_0(x)$, $u_1(x)$. This describes e.g. a rectangular elastic membrane which is fixed at the boundary. Here $u(x)$ is the displacement of the membrane.

2.3 Poisson equation on a rectangle

Now we consider a given function $f(x, y)$. We want to find a function $u(x, y)$ for $(x, y) \in G$ such that

$$-\Delta u(x, y) = f(x, y) \quad (x, y) \in G \quad (15)$$

$$u(x, y) = 0 \quad (x, y) \in \partial G \quad (16)$$

This describes the equilibrium problem for either the heat equation of the wave equation, i.e., temperature in a rectangular plate at equilibrium, or displacement of a rectangular membrane at equilibrium.

2.4 Eigenvalue problem for Laplace operator on a rectangle

For all three problems (heat equation, wave equation, Poisson equation) we first have to solve an **eigenvalue problem**: Find functions $v(x, y)$ and numbers λ such that

$$-\Delta v(x, y) = \lambda v(x, y) \quad (x, y) \in G \quad (17)$$

$$v(x, y) = 0, \quad (x, y) \in \partial G \quad (18)$$

We will always have $\lambda \geq 0$. In the case Neumann conditions on the whole boundary there is an eigenvalue $\lambda = 0$, in all other cases (as in the Dirichlet case here) we have $\lambda > 0$.

In order to solve the eigenvalue problem we use separation of variables and try to find eigenfunctions of the form

$$v(x, y) = p(x)q(y)$$

The function $v(x, y)$ must satisfy the boundary condition (18), so we must have

$$p(0) = 0, \quad p(L_1) = 0, \quad q(0) = 0, \quad q(L_2) = 0 \quad (19)$$

We plug $v(x, y) = p(x)q(y)$ into (17):

$$\begin{aligned} -v_{xx}(x, y) - v_{yy}(x, y) &= -p''(x)q(y) - p(x)q''(y) = \lambda p(x)q(y) \\ -\frac{p''(x)}{p(x)} - \frac{q''(y)}{q(y)} &= \lambda \end{aligned}$$

Since $-\frac{p''(x)}{p(x)}$ depends only on x and $-\frac{q''(y)}{q(y)}$ depends only on y both terms must be constants:

$$-\frac{p''(x)}{p(x)} = \mu, \quad -\frac{q''(y)}{q(y)} = \nu, \quad \mu + \nu = \lambda \quad (20)$$

Therefore we obtain two eigenvalue problems for $p(x)$ and $q(y)$:

1. Find $p(x)$ and μ such that

$$-p''(x) = \mu p(x), \quad p(0) = 0, \quad p(L_1) = 0 \quad (21)$$

This gives the eigenvalues $\mu_j = \left(\frac{j\pi}{L_1}\right)^2$ and eigenfunctions $p_j(x) = \sin\left(\frac{j\pi}{L_1}x\right)$ for $j = 1, 2, \dots$. Note that the eigenfunctions are orthogonal

$$\int_0^{L_1} p_j(x)p_k(x)dx = 0 \quad \text{for } j \neq k$$

and complete on the interval $[0, L_1]$.

2. Find $q(y)$ and ν such that

$$-q''(y) = \nu q(y), \quad q(0) = 0, \quad q(L_2) = 0 \quad (22)$$

This gives the eigenvalues $\nu_k = \left(\frac{k\pi}{L_2}\right)^2$ and eigenfunctions $q_k(y) = \sin\left(\frac{k\pi}{L_2}y\right)$ for $k = 1, 2, \dots$. Note that the eigenfunctions are orthogonal

$$\int_0^{L_2} q_j(y)q_k(y)dy = 0 \quad \text{for } j \neq k$$

and complete on the interval $[0, L_2]$.

Therefore we have found eigenfunctions $v_{jk}(x, y) = p_j(x)q_k(y)$ and eigenvalues $\lambda_{jk} = \mu_j + \nu_k$ for $j = 1, 2, \dots, k = 1, 2, \dots$.

We claim that the eigenfunctions have the following properties:

- **orthogonality:** With the **inner product** $\langle f, g \rangle := \iint_G f(x, y)g(x, y)dxdy$ we have

$$\langle v_{jk}, v_{j'k'} \rangle = 0 \quad \text{for } (j, k) \neq (j', k')$$

This is easy to see:

$$\int_{x=0}^{L_1} \int_{y=0}^{L_2} v_{jk}(x, y)v_{j'k'}(x, y)dxdy = \int_{x=0}^{L_1} p_j(x)p_{j'}(x)dx \int_{y=0}^{L_2} q_k(y)q_{k'}(y)dy$$

If $j \neq j'$ the first term on the right hand side is zero. If $k \neq k'$ the second term on the right hand side is zero.

- **completeness:** For any $F(x, y)$ where $\iint_G F(x, y)^2 dxdy$ exists we can define the **Fourier coefficients**

$$F_{jk} := \frac{\langle F, v_{jk} \rangle}{\langle v_{jk}, v_{jk} \rangle} = \frac{\int_{x=0}^{L_1} \int_{y=0}^{L_2} F(x, y)p_j(x)q_k(y)dydx}{\left(\int_{x=0}^{L_1} p_j(x)^2 dx\right) \left(\int_{y=0}^{L_2} q_k(y)^2 dy\right)}$$

and then have that the **Fourier series** converges to $F(x, y)$

$$F(x, y) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} F_{jk} v_{jk}(x, y)$$

in the sense that $\left\| F - \sum_{j=1}^N \sum_{k=1}^N F_{jk} v_{jk} \right\| \rightarrow 0$ as $N \rightarrow \infty$. Here $\|g\| := \langle g, g \rangle^{1/2}$.

2.5 Solution of the Poisson problem on a rectangle

We can represent the solution $u(x, y)$ in terms of its Fourier series

$$u(x, y) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} u_{jk} v_{jk}(x, y)$$

where we need to find the coefficients u_{jk} . Therefore we plug this into the ODE $-\Delta u(x, y) = f(x, y)$

$$-\Delta u(x, y) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} u_{jk} (-\Delta v_{jk}(x, y)) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} u_{jk} \lambda_{jk} v_{jk}(x, y) \stackrel{!}{=} f(x, y)$$

and obtain that $u_{jk} \lambda_{jk}$ are the Fourier coefficients of the function $f(x)$, i.e.,

$$u_{jk} := \lambda_{jk}^{-1} \frac{\langle f, v_{jk} \rangle}{\langle v_{jk}, v_{jk} \rangle}$$

2.6 Solution of the heat equation on a rectangle

We first consider the heat equation $u_t - \Delta u = 0$ with $f(x, y, t) = 0$. The separation of variables method means that we first look for special solutions of the form $u(x, y, t) = g(t)v(x, y)$ and obtain

$$u_t - \Delta u = g'(t)v(x, y) - g(t)\Delta v(x, y) = 0$$

$$-\frac{g'(t)}{g(t)} = -\frac{\Delta v(x, y)}{v(x, y)} = \lambda$$

$$-\Delta v''(x, y) = \lambda v(x, y) \quad \text{for } (x, y) \in G, \quad v(x, y) = 0 \quad \text{for } (x, y) \in \partial G \quad (23)$$

$$g'(t) + \lambda g(t) = 0 \quad (24)$$

Equation (23) means that $v(x, y)$ is an eigenfunction $v_{jk}(x, y)$ and λ is an eigenvalue λ_{jk} , $j = 1, 2, \dots$, $k = 1, 2, \dots$. Equation (24) is an ODE with general solution

$$g(t) = C_{jk} e^{-\lambda_{jk} t}.$$

Therefore the special solutions with separated variables are

$$C_{jk} e^{-\lambda_{jk} t} v_{jk}(x, y), \quad j = 1, 2, \dots, \quad k = 1, 2, \dots$$

For the solution of our initial value problem $u_t - \Delta u = 0$ with initial condition $u(x, y, 0) = u_0(x, y)$ we want to write the solution as a linear combination of these special solutions:

$$u(x, y, t) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} C_{jk} e^{-\lambda_{jk} t} v_{jk}(x, y)$$

This will satisfy the PDE, but we also need to satisfy the initial condition:

$$u(x, y, 0) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} C_{jk} v_{jk}(x, y) \stackrel{!}{=} u_0(x, y)$$

Therefore C_{jk} must be the Fourier coefficients of the function $u_0(x, y)$:

$$C_{jk} := \frac{\langle u_0, v_{jk} \rangle}{\langle v_{jk}, v_{jk} \rangle} \quad (25)$$

For the general problem $u_t - \Delta u = f(x, y, t)$ with a source function $f(x, y, t)$ we can then use Duhamel's principle:

$$\begin{aligned}
u(x, y, t) &= u_{\text{hom}}(x, y, t) + u_{\text{part}}(x, y, t) \\
u_{\text{hom}}(x, y, t) &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} C_{jk} e^{-\lambda_{jk} t} v_{jk}(x, y), \quad C_{jk} \text{ given by (25)} \\
u_{\text{part}}(x, y, t) &= \int_{s=0}^t z_{(s)}(x, y, t) ds \\
z_{(s)}(x, y, t) &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} F_{jk}(s) e^{-\lambda_{jk}(t-s)} v_{jk}(x, y), \quad F_{jk}(s) := \frac{\langle f(\cdot, \cdot, s), v_{jk} \rangle}{\langle v_{jk}, v_{jk} \rangle}
\end{aligned}$$

2.7 Solution of the wave equation on a rectangle

We first consider the wave equation $u_{tt} - \Delta u = 0$ with $f(x, y, t) = 0$. The separation of variables method means that we first look for special solutions of the form $u(x, y, t) = g(t)v(x, y)$ and obtain

$$\begin{aligned}
u_{tt} - \Delta u &= g''(t)v(x) - g(t)\Delta v(x, y) = 0 \\
-\frac{g''(t)}{g(t)} &= -\frac{\Delta v(x, y)}{v(x, y)} = \lambda \\
-\Delta v''(x, y) &= \lambda v(x, y) \quad \text{for } (x, y) \in G, \quad v(x, y) = 0 \quad \text{for } (x, y) \in \partial G \quad (26) \\
g''(t) + \lambda g(t) &= 0 \quad (27)
\end{aligned}$$

Equation (26) means that $v(x, y)$ is an eigenfunction $v_{jk}(x, y)$ and λ is an eigenvalue $\lambda_{j,k}$, $j = 1, 2, \dots$, $k = 1, 2, \dots$. Equation (27) is for $\lambda_{jk} > 0$ an ODE with general solution

$$g(t) = A_{jk} \cos(\lambda_{jk}^{1/2} t) + B_{jk} \sin(\lambda_{jk}^{1/2} t)$$

Therefore the special solutions with separated variables are

$$\left[A_{jk} \cos(\lambda_{jk}^{1/2} t) + B_{jk} \sin(\lambda_{jk}^{1/2} t) \right] v_{jk}(x, y), \quad j = 1, 2, \dots, \quad k = 1, 2, \dots$$

For the solution of our initial value problem $u_t - \Delta u = 0$ with initial conditions $u(x, y, 0) = u_0(x, y)$ and $u_t(x, y, 0) = u_1(x, y, t)$ we want to write the solution as a linear combination of these special solutions:

$$\boxed{u(x, y, t) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left[A_{jk} \cos(\lambda_{jk}^{1/2} t) + B_{jk} \sin(\lambda_{jk}^{1/2} t) \right] v_{jk}(x, y)} \quad (28)$$

This will satisfy the PDE, but we also need to satisfy the initial conditions:

$$\begin{aligned}
u(x, y, 0) &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} A_{jk} v_{jk}(x, y) \stackrel{!}{=} u_0(x, y) \\
u_t(x, y, 0) &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} B_{jk} \lambda_{jk}^{1/2} v_{jk}(x, y) \stackrel{!}{=} u_1(x, y)
\end{aligned}$$

Therefore A_{jk} must be the Fourier coefficients of the function $u_0(x, y)$, and $\lambda_{jk}^{1/2} B_{jk}$ must be the Fourier coefficients of the function $u_1(x, y)$:

$$\boxed{A_{jk} := \frac{\langle u_0, v_{jk} \rangle}{\langle v_{jk}, v_{jk} \rangle}, \quad B_{jk} := \lambda_{jk}^{-1/2} \frac{\langle u_1, v_{jk} \rangle}{\langle v_{jk}, v_{jk} \rangle}} \quad (29)$$

For the general problem $u_{tt} - \Delta u = f(x, y, t)$ with a source function $f(x, y, t)$ we can then use Duhamel's principle:

$$\begin{aligned} u(x, y, t) &= u_{\text{hom}}(x, y, t) + u_{\text{part}}(x, y, t) \\ u_{\text{hom}}(x, y, t) &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left[A_{jk} \cos(\lambda_{jk}^{1/2} t) + B_{jk} \sin(\lambda_{jk}^{1/2} t) \right] v_{jk}(x), \quad A_{jk}, B_{jk} \text{ given by (29)} \\ u_{\text{part}}(x, y, t) &= \int_{s=0}^t z_{(s)}(x, y, t) ds \\ z_{(s)}(x, y, t) &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} F_{jk}(s) \sin(\lambda_{jk}^{1/2}(t-s)) v_{jk}(x, y), \quad F_{jk}(s) := \lambda_{jk}^{-1/2} \frac{\langle f(\cdot, \cdot, s), v_{jk} \rangle}{\langle v_{jk}, v_{jk} \rangle} \end{aligned}$$

2.8 Examples on a rectangle

We consider the rectangle $G = [0, 2] \times [0, 1]$ with Dirichlet conditions on all four sides.

2.8.1 Example for Poisson equation on a rectangle

We consider the Poisson problem (15), (16) with the function $f(x, y) = 1$.

We first have to solve the eigenvalue problem (17), (18) on a rectangle: Solving the eigenvalue problem (21) gives with $L_1 = 2$

$$\mu_j = \left(\frac{j\pi}{2} \right)^2, \quad p_j(x) = \sin\left(\frac{j\pi}{2} x \right) \quad j = 1, 2, \dots$$

Solving the eigenvalue problem (22) gives with $L_2 = 1$

$$v_k = (k\pi)^2, \quad q_k(y) = \sin(k\pi y) \quad k = 1, 2, \dots$$

Therefore we obtain for $j = 1, 2, \dots$ and $k = 1, 2, \dots$ the eigenvalues λ_{jk} and eigenfunctions

$$\lambda_{jk} = \left(\frac{j^2}{4} + k^2 \right) \pi^2, \quad v_{jk}(x, y) = \sin\left(\frac{j\pi}{2} x \right) \sin(k\pi y).$$

We have using (1) that

$$\langle v_{jk}, v_{jk} \rangle = \left(\int_{x=0}^{L_1} p_j(x)^2 dx \right) \left(\int_{y=0}^{L_2} q_k(y)^2 dy \right) = \frac{L_1}{2} \cdot \frac{L_2}{2} = \frac{2}{2} \cdot \frac{1}{2} = \frac{1}{2}. \quad (30)$$

Next we need to compute $\langle f, v_{jk} \rangle$:

$$\langle f, v_{jk} \rangle = \langle 1, v_{jk} \rangle = \int_{x=0}^2 \int_{y=0}^1 1 \cdot p_j(x) \cdot q_k(y) dy dx = \left(\int_{x=0}^2 \sin\left(\frac{j\pi}{2} x \right) dx \right) \left(\int_{y=0}^1 \sin(k\pi y) dy \right)$$

where

$$\int_{x=0}^L \sin\left(\frac{j\pi}{L} x \right) dx = \frac{L}{j\pi} \int_{z=0}^{j\pi} \sin(z) dz = \frac{L}{j\pi} [-\cos(z)]_{z=0}^{j\pi} = \frac{L}{j\pi} \begin{cases} 2 & \text{for } j \text{ odd} \\ 0 & \text{for } j \text{ even} \end{cases}$$

yielding

$$\langle 1, v_{jk} \rangle = \begin{cases} \frac{4}{j\pi} \cdot \frac{2}{k\pi} & \text{if both } j, k \text{ are odd} \\ 0 & \text{otherwise} \end{cases} \quad (31)$$

Therefore we obtain $u(x, y) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} u_{jk} v_{jk}(x, y)$ where for both j, k odd we have

$$u_{jk} := \lambda_{jk}^{-1} \frac{\langle f, v_{jk} \rangle}{\langle v_{jk}, v_{jk} \rangle} = \frac{1}{\pi^2 \left(\frac{j^2}{4} + k^2 \right)} \cdot \frac{\frac{8}{jk\pi^2}}{\frac{1}{2}} = \frac{16}{\pi^4 jk \left(\frac{j^2}{4} + k^2 \right)},$$

otherwise $u_{jk} = 0$. Therefore we can write the solution as

$$u(x, y) = \sum_{j=1,3,5,\dots} \sum_{k=1,3,5,\dots} \frac{16}{\pi^4 jk \left(\frac{j^2}{4} + k^2\right)} \sin\left(\frac{j\pi}{2}x\right) \sin(k\pi y).$$

2.8.2 Example for heat equation on a rectangle

We consider the heat equation (8)–(10) with $f(x, y, t) = 0$ and $u_0(x, y) = 1$. We already know the eigenvalues λ_{jk} and eigenfunctions $v_{jk}(x, y)$. We then have to find the Fourier coefficients of $u_0(x, y)$:

$$C_{jk} := \frac{\langle u_0, v_{jk} \rangle}{\langle v_{jk}, v_{jk} \rangle} = \frac{\langle 1, v_{jk} \rangle}{\langle v_{jk}, v_{jk} \rangle} \quad (32)$$

From (30), (31) we get

$$C_{jk} = \begin{cases} \frac{16}{jk\pi^2} & \text{if both } j, k \text{ are odd} \\ 0 & \text{otherwise} \end{cases} \quad (33)$$

yielding

$$u(x, y, t) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} C_{jk} e^{-\lambda_{jk}t} v_{jk}(x, y) = \sum_{j=1,3,5,\dots} \sum_{k=1,3,5,\dots} \frac{16}{jk\pi^2} \exp\left(-\left(\frac{j^2}{4} + k^2\right)\pi^2 t\right) \sin\left(\frac{j\pi}{2}x\right) \sin(k\pi y).$$

This may look complicated. But it shows that the different eigenmodes have different decay rates: For large frequencies j, k we have large λ_{jk} and fast decay. For a large time t the solution is dominated by the term with $j = 1$ and $k = 1$:

$$u(x, y, t) \approx \frac{16}{\pi^2} \exp\left(-\frac{5}{4}\pi^2 t\right) \sin\left(\frac{\pi}{2}x\right) \sin(\pi y).$$

2.8.3 Example for wave equation on a rectangle

We consider the wave equation (11)–(14) with $f(x, y, t) = 0$, $u_0(x, y) = 0$ and $u_1(x, y) = 1$. We already know the eigenvalues λ_{jk} and eigenfunctions $v_{jk}(x, y)$. We then have $A_{jk} = 0$ and we have to find the coefficients B_{jk} : We can use (32), (33) and obtain

$$B_{jk} = \lambda_{jk}^{-1/2} \frac{\langle u_1, v_{jk} \rangle}{\langle v_{jk}, v_{jk} \rangle} = \lambda_{jk}^{-1/2} \frac{\langle 1, v_{jk} \rangle}{\langle v_{jk}, v_{jk} \rangle} = \left[\left(\frac{j^2}{4} + k^2\right)\pi^2\right]^{-1/2} C_{jk} = \begin{cases} \frac{16}{\left(\frac{j^2}{4} + k^2\right)^{1/2} jk\pi^3} & \text{if both } j, k \text{ are odd} \\ 0 & \text{otherwise} \end{cases}$$

Then (28) gives

$$u(x, y, t) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} B_{jk} \sin(\lambda_{jk}^{1/2} t) v_{jk}(x, y) = \sum_{j=1,3,5,\dots} \sum_{k=1,3,5,\dots} \frac{16}{\left(\frac{j^2}{4} + k^2\right)^{1/2} jk\pi^3} \sin\left(\left(\frac{j^2}{4} + k^2\right)^{1/2} \pi t\right) \sin\left(\frac{j\pi}{2}x\right) \sin(k\pi y)$$

We see that the eigenmodes with higher spatial frequencies j, k oscillate with higher eigenfrequency $\lambda_{jk}^{1/2}$ in time. Note that unlike the case of the one-dimensional string the higher time eigenfrequencies $\lambda_{jk}^{1/2}$ are no longer integer multiples of the lowest eigenfrequency $\lambda_{11}^{1/2} = \left(\frac{5}{4}\right)^{1/2} \pi$. Therefore the sound generated by the vibrating membrane is not a periodic function.