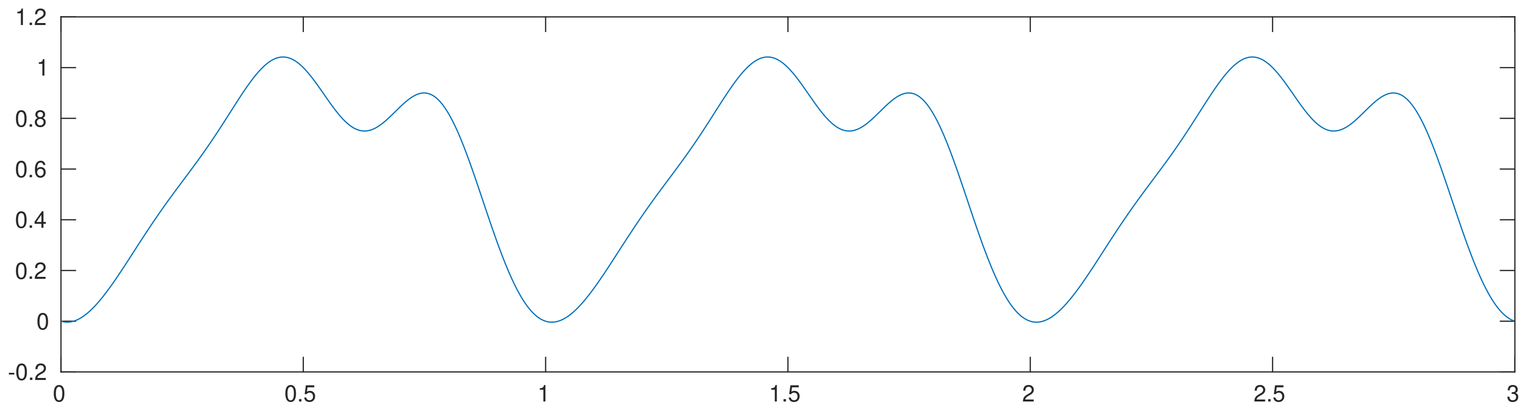
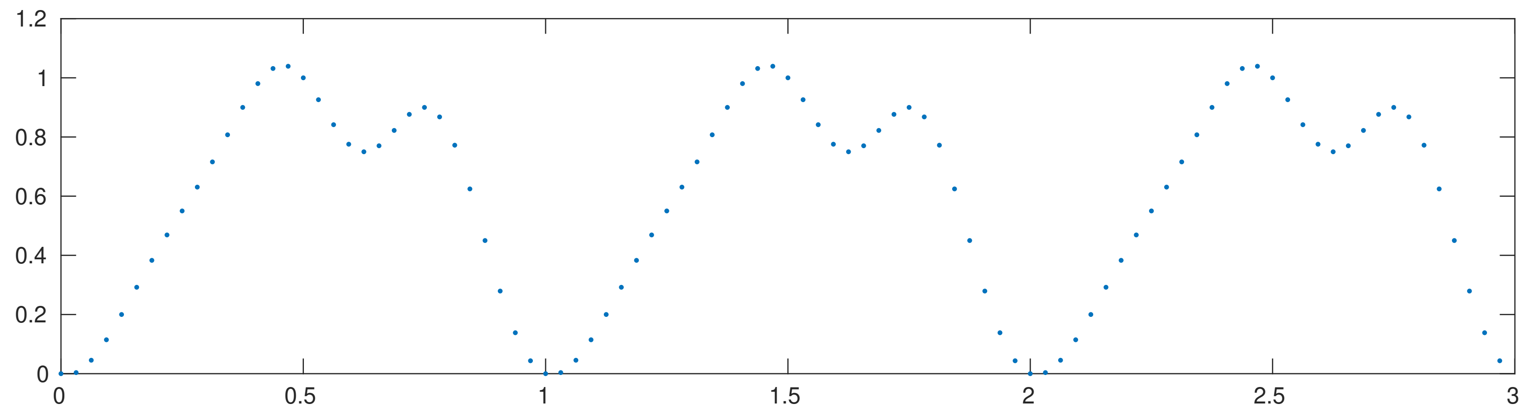
1 Introduction to Fourier analysis

We consider a signal $f(x)$ where x is time.

This may be a sound recorded my a microphone, and give something like this:



In practice we usually are only given function values $f(x_j)$ for points $x_j = jh$ with a step size h ("sampling"), e.g., for sound recorded on a CD we have $h = \frac{1}{44100}$ sec, here $h = \frac{1}{32}$:



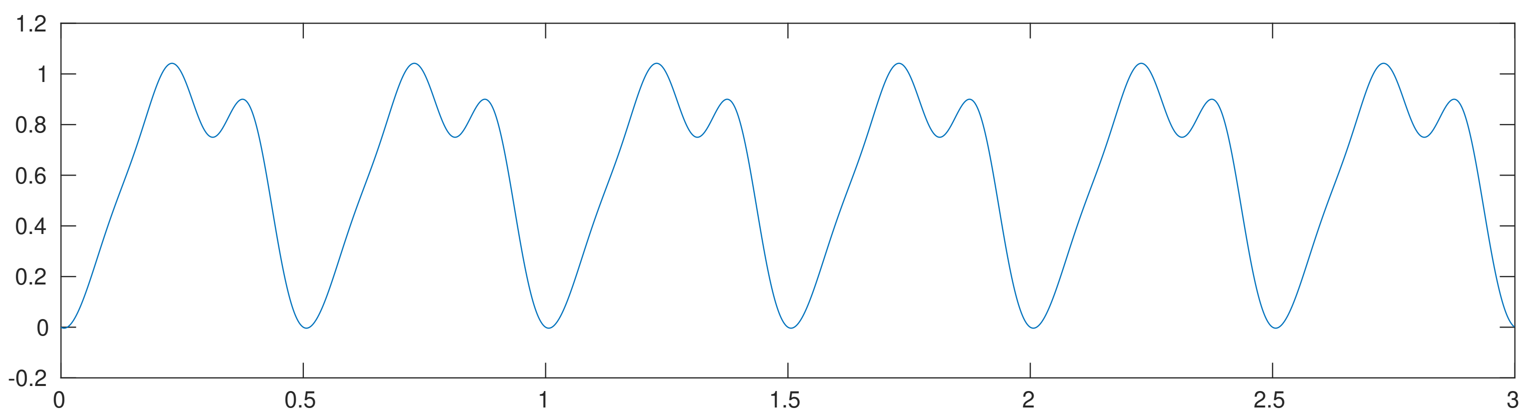
Describing a signal in terms of function values is called a representation in the **time domain**.

Note that our sample signal $f(x)$ is **periodic**: We have for all $x \in \mathbb{R}$

$$f(x+L) = f(x)$$

with the **period** L. Here $L = 1$.

Consider the signal $g(x) := f(2x)$:



Here the period is $L = \frac{1}{2}$. The **frequency** is $\xi = 1/L$, this measures how many periods we have on a unit interval (ξ can be noninteger, e.g., for $f(2.5x)$ we have $\xi = 2.5$).

When we hear a sound signal we can directly perceive frequencies, but not sample values in the time domain.

Consider a signal with period $L = 1$, like our sample signal.

What functions with $f(x) = f(x+1)$ do we know? The functions $\cos(2\pi x)$ and $\sin(2\pi x)$ work, but so do

$$\cos(2\pi kx), \qquad \sin(2\pi kx)$$

with integer frequencies $k \in \{0,1,2,3,\ldots\}$. For $k = 0$ we only have $\cos(2\pi \cdot 0 \cdot x) = 1$ which is constant.

The idea of Fourier analysis is to write the signal $f(x)$ as a superposition of these functions.

Since

$$e^{ix} = \cos(x) + i\sin(x),$$
$$\cos(x) = \tfrac{1}{2}\left(e^{ix} + e^{-ix}\right), \qquad \sin(x) = \tfrac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

we can use instead of the functions

$$\begin{array}{l} 1, \quad \cos(2\pi x), \quad \cos(4\pi x), \quad \cos(6\pi x), \quad \ldots \\ \qquad \sin(2\pi x), \quad \sin(4\pi x), \quad \sin(6\pi x), \quad \ldots \end{array}$$

the functions

$$\ldots, e^{-i6\pi x}, e^{-i4\pi x}, e^{-i2\pi x}, 1, e^{i2\pi x}, e^{i4\pi x}, e^{i6\pi x}, \ldots$$

i.e., we want to write our 1-periodic signal $f(x)$ as

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{i2\pi kx}$$

where $\hat{f}_k \in \mathbb{C}$ are the so-called **Fourier coefficients** and describe our signal in the "**frequency domain**".

Example: For the signal $f(x) = \sin(2\pi x)$ we have $f(x) = \frac{1}{2i}\left(e^{2\pi ix} - e^{-2\pi ix}\right)$, hence the Fourier coefficients are $\hat{f}_{-1} = \frac{1}{2}i$, $\hat{f}_1 = -\frac{1}{2}i$, and $\hat{f}_k = 0$ for all $k \in \mathbb{Z} \setminus \{-1,1\}$. We see that a real-valued signal may have complex, non-real Fourier coefficients.

We will also consider **nonperiodic signals** $f(x)$. In this case we will need to use functions $e^{i2\pi\xi x}$ with frequencies $\xi \in \mathbb{R}$, and we will write our signal as

$$f(x) = \int_{\xi=-\infty}^{\infty} \hat{f}(\xi) e^{i2\pi\xi x}$$

The **Fourier transform** takes us from the time domain to the frequency domain. The **inverse Fourier transform** takes us from the frequency domain to the time domain.

We will consider **four different settings** for the "Fourier transform":

	time domain	frequency domain
1-periodic function, continuous time	$f(x)$, $x \in [0,1)$	$\hat{f}_k$, $k \in \mathbb{Z}$
1-periodic function, discrete time	f_j for $x_j = \frac{j}{N}$, $j = 0,\ldots,N-1$	$\hat{f}_k$, $k = 0,\ldots,N-1$
Nonperiodic function, continuous time	$f(x)$, $x \in \mathbb{R}$	$\hat{f}(\xi)$, $\xi \in \mathbb{R}$
Nonperiodic function, discrete time	f_j for $x_j = jh$, $j \in \mathbb{Z}$	$\hat{f}(\xi)$, $\xi \in \left[-\frac{1}{2h}, \frac{1}{2h}\right]$

Warning: There are different conventions for defining "Fourier transforms" in use in books and mathematical software: Some people use $e^{i\xi x}$ instead of $e^{2\pi i\xi x}$, and different factors 2π or $\sqrt{2\pi}$ are used.

Mathematically speaking, all these definitions are equivalent, as one can go from one convention to a different one by inserting factors 2π or $\sqrt{2\pi}$ in appropriate places.

In practice one has to be careful e.g. when using software like Matlab, Mathematica, Maple: You have to check the precise definition of "Fourier transform" which the software uses.

In this class I will **always represent** $f(x)$ **as a superposition of terms** $\hat{f}(\xi)\cdot e^{2\pi i\xi x}$ **or** $\hat{f}_k\cdot e^{2\pi ikx}$. I will explain how to use **Matlab** which uses different definitions.

We will first consider functions with period 1, i.e., $f(x+1)=f(x)$ for all x. Once we understand this case, it is easy to generalize this to **functions with period** L**, i.e.,** $f(x+L)=f(x)$**:**

	time domain	frequency domain
L-periodic function, continuous time	$f(x)$, $x\in[0,L)$	$\hat{f}_k$ for $\xi_k=\frac{k}{L}$, $k\in\mathbb{Z}$
L-periodic function, discrete time	f_j for $x_j=\frac{j}{N}L$, $j=0,\dots,N-1$	$\hat{f}_k$ for $\xi_k=\frac{k}{L}$, $k=0,\dots,N-1$

2 Linear algebra: vectors, subspaces, orthogonal basis, orthogonal projection

2.1 Motivation: Fourier series

We are given a 1-periodic function $f(x)$. We want to write this in terms of the functions $u^{(k)}(x):=e^{2\pi ikx}$:

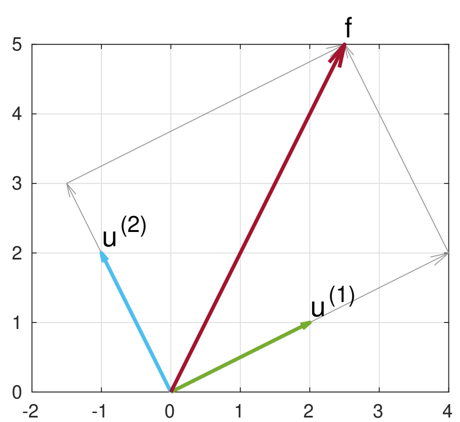
$$f=\sum_{k=-\infty}^{\infty}\hat{f}_k u^{(k)}$$

This means we have to find the Fourier coefficients $\hat{f}_k\in\mathbb{C}$ for all $k\in\mathbb{Z}$.

It turns out that this is a linear algebra problem, similar to the following:

Example problem in the vector space $V=\mathbb{R}^2$: We are given the vector $f=\begin{bmatrix}2.5\\5\end{bmatrix}$, and we want to write this as a linear combination of the vectors $u^{(1)}=\begin{bmatrix}2\\1\end{bmatrix}$ and $u^{(2)}=\begin{bmatrix}-1\\2\end{bmatrix}$: Find $c_1,c_2\in\mathbb{R}$ such that

$$f=c_1u^{(1)}+c_2u^{(2)}$$



Solution using results from section 2.3: As $\left(u^{(1)},u^{(2)}\right)=0$ the vectors $u^{(1)},u^{(2)}$ are an orthogonal basis of V, hence

$$c_1=\frac{\left(f,u^{(1)}\right)}{\left(u^{(1)},u^{(1)}\right)}=\frac{10}{5}=2,\qquad c_2=\frac{\left(f,u^{(2)}\right)}{\left(u^{(2)},u^{(2)}\right)}=\frac{7.5}{5}=1.5$$

2.2 Vector spaces, span, inner product

"Vector space" means that we specify a set of **vectors**, and a set of **scalars**. We can add two vectors, and we can multiply a vector by a scalar.

Example: $V = \mathbb{R}^3$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ with $x_1, x_2, x_3 \in \mathbb{R}^3$. Scalars are numbers in $\mathbb{R}$.

Example: $V = \mathbb{C}^n$ is a vector space: vectors have the form $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ with $u_j \in \mathbb{C}$, and scalars are numbers in $\mathbb{C}$.

For vectors $u^{(1)}, \dots, u^{(N)}$ and scalars $c_1, \dots, c_N$ the **linear combination** $v = c_1 u^{(1)} + \cdots + c_N u^{(N)}$ is again a vector.

If $c_1 u^{(1)} + \cdots + c_N u^{(N)}$ can only be the zero vector for $c_1 = \cdots = c_N = 0$ we say that the vectors $u^{(1)}, \dots, u^{(N)}$ are **linearly independent**.

We denote the set of all linear combinations by

$$\operatorname{span}\left\{u^{(1)}, \dots, u^{(N)}\right\} = \left\{c_1 u^{(1)} + \cdots + c_N u^{(N)} \mid c_j \in \mathbb{C}\right\}$$

$W = \operatorname{span}\left\{u^{(1)}, \dots, u^{(N)}\right\}$ is a **subspace** of V: W is a vector space with $W \subset V$.

Example: $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \right\}$ is the x_1, x_2-plane in $\mathbb{R}^3$. Note that one of the 3 vectors is "redundant".

If $W = \operatorname{span}\left\{u^{(1)}, \dots, u^{(N)}\right\}$ and the vectors $u^{(1)}, \dots, u^{(N)}$ are linearly independent:

- the vectors $u^{(1)}, \dots, u^{(N)}$ are called a **basis** of the subspace W
- N is called the **dimension** of the subspace W

For the vector space $\mathbb{C}^n$ we introduce the **inner product**

$$(u, v) = \sum_{i=1}^{n} u_i \bar{v}_i$$

Here $\bar{v}_i$ denotes the *complex conjugate*: For a complex number $z = x + iy$ we have $\bar{z} = x - iy$.

The **norm** of a vector is defined as

$$\|u\| = (u, u)^{1/2}. \tag{1}$$

For a vector $u \in \mathbb{C}^n$ this means that $\|u\| = \sqrt{|u_1|^2 + \cdots + |u_n|^2}$.

We say that two vectors u, v are **orthogonal** if $(u, v) = 0$.

An important property is the **Cauchy-Schwarz inequality**:

$$|(u, v)| \leq \|u\| \, \|v\| \tag{2}$$

In (1) we defined the norm in terms of the inner product. Conversely, we can also express the inner product (u, v) in terms of norms $\|\cdot\|$:

For real vectors $u, v \in \mathbb{R}^n$ we have

$$\|u+v\|^2 - \|u-v\|^2 = \left(\|u\|^2 + 2(u,v) + \|v\|^2\right) - \left(\|u\|^2 - 2(u,v) + \|v\|^2\right) = 4(u,v)$$
$$(u, v) = \frac{1}{4}\left[\|u+v\|^2 - \|u-v\|^2\right]$$

For complex vectors $u, v \in \mathbb{C}^n$ we can use the identity

$$(u, v) = \frac{1}{2}\left[\|u+v\|^2 - i\,\|u+iv\|^2 - (1-i)\,\|u\|^2 - (1-i)\,\|v\|^2\right] \tag{3}$$

2.3 Orthogonal projection on a subspace W

If the nonzero vectors $u^{(1)},\dots,u^{(N)}$ are orthogonal on each other, i.e.,

$$\left(u^{(k)},u^{(l)}\right)=0 \qquad \text{for } k\neq l$$

we say that they form an **orthogonal basis** of the subspace

$$W=\operatorname{span}\left\{u^{(1)},\dots,u^{(N)}\right\}.$$

For a given vector v we now want to find the vector $w\in W$ which is closest to w, i.e., $\|v-w\|$ should be minimal. In this case we must have that the difference vector $w-v$ is orthogonal on the subspace W, i.e.,

$$\left(v-w,u^{(k)}\right)=0 \qquad \text{for } k=1,\dots,N. \tag{4}$$

We want to find the coefficients $c_1,\dots,c_N$ of the vector $v=c_1u^{(1)}+\cdots+c_Nu^{(N)}$. Plugging this into (4) gives using the orthogonality of the vectors $u^{(1)},\dots,u^{(N)}$ that

$$\boxed{c_k=\frac{\left(v,u^{(k)}\right)}{\left(u^{(k)},u^{(k)}\right)}} \tag{5}$$

For a given vector w we can find the projection v as follows:

1. Compute the coefficients $c_k=\frac{\left(v,u^{(k)}\right)}{\left(u^{(k)},u^{(k)}\right)}$ for $k=1,\dots,N$
2. Let $w=c_1u^{(1)}+\cdots+c_Nu^{(N)}$.

Claim: This vector w gives the minimal error $\|v-w\|$ among all $w\in W$:
Proof: Assume that we use a vector $w+d$ with some $d\in W$ instead of w:

$$\|v-(w+d)\|^2=(v-w-d,v-w-d)=(v-w,v-w)-\underbrace{(v-w,d)}_{0}-\underbrace{(d,v-w)}_{0}+(d,d)$$

as $v-w$ is orthogonal on all vectors in W. Hence we have for nonzero d

$$\|v-(w+d)\|^2=\|v-w\|^2+\underbrace{\|d\|^2}_{0}>\|v-w\|^2, \tag{6}$$

i.e., the error of $v+d$ is strictly larger than the error of v.□

Since $v-w$ is orthogonal on w we have the Pythagoras equation

$$(v,v)=(w+(v-w),w+(v-w))=(w,w)+\underbrace{(v-w,w)}_{0}+\underbrace{(w,v-w)}_{0}+(v-w,v-w)$$

$$\|v\|^2=\|w\|^2+\|v-w\|^2$$

As $\|v-w\|\geq 0$ we obviously have

$$\|v\|\geq\|w\|$$

Since $w=c_1u^{(1)}+\cdots+c_Nu^{(N)}$ with orthogonal vectors $u^{(1)},\dots,u^{(N)}$ we have

$$\|w\|^2=|c_1|^2\left\|u^{(1)}\right\|^2+\cdots+|c_N|^2\left\|u^{(N)}\right\|^2 \tag{7}$$

and hence

$$\boxed{\|v\|^2=|c_1|^2\left\|u^{(1)}\right\|^2+\cdots+|c_N|^2\left\|u^{(N)}\right\|^2+\|v-w\|^2} \tag{8}$$

This equation can be used to compute $\|v-w\|$. Since $\|v-w\|\geq 0$ we obviously have

$$\|v\|^2\geq|c_1|^2\left\|u^{(1)}\right\|^2+\cdots+|c_N|^2\left\|u^{(N)}\right\|^2. \tag{9}$$

2.4 Example for $\mathbb{R}^3$

Consider the vectors $u^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $u^{(2)} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ and let $W := \operatorname{span}\{u^{(1)}, u^{(2)}\}$. Note that $\left(u^{(1)}, u^{(2)}\right) = 0$, hence $u^{(1)}, u^{(2)}$ are an orthogonal basis of W, and W has dimension 2. Note that $W \subset V = \mathbb{R}^3$ which is a vector space of dimension 3.

Now let $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. We want to find the closest vector $w \in W$:

$$c_1 = \frac{\left(v, u^{(1)}\right)}{\left(u^{(1)}, u^{(1)}\right)} = \frac{4}{2} = 2, \qquad c_2 = \frac{\left(v, u^{(2)}\right)}{\left(u^{(2)}, u^{(2)}\right)} = \frac{4}{3}, \qquad w = c_1 u^{(1)} + c_2 u^{(2)} = \frac{1}{3}\begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$$

We can check that $v - w$ is orthogonal on both $u^{(1)}, u^{(2)}$: $v - w = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, so $\left(v - w, u^{(1)}\right) = 0$, $\left(v - w, u^{(2)}\right) = 0$.

We can compute the distance of the point v to the subspace W in two ways:

1. $\|v - w\| = \left\| \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\| = \left[\frac{1}{9} \cdot (1 + 4 + 1)\right]^{1/2} = \sqrt{\frac{2}{3}}$
2. $\|v - w\|^2 = \|v\|^2 - |c_1|^2 \left\|u^{(1)}\right\|^2 - |c_2|^2 \left\|u^{(2)}\right\|^2 = 14 - 2^2 \cdot 2 - (\frac{4}{3})^2 \cdot 3 = \frac{2}{3}$

2.5 Orthonormal basis $u^{(1)}, \dots, u^{(N)}$

If the orthogonal vectors $u^{(1)}, \dots, u^{(N)}$ have all length 1, i.e.

$$\left(u^{(k)}, u^{(l)}\right) = \begin{cases} 0 & \text{for } k \neq l \\ 1 & \text{for } k = l \end{cases}$$

some of the formulas from the previous section become simpler:

The coefficients c_k of the projection $v = c_1 u^{(1)} + \cdots + c_N u^{(N)}$ are given by

$$c_k = \left(v, u^{(k)}\right) \tag{10}$$

and (8), (9) become

$$\|v\|^2 = |c_1|^2 + \cdots + |c_N|^2 + \|v - w\|^2 \tag{11}$$

$$\|v\|^2 \geq |c_1|^2 + \cdots + |c_N|^2 \tag{12}$$

2.6 Functions as vectors

Sets of functions can also form a vector space. For example,

let V denote the **set of all complex-valued continuous functions on the interval** $[0, 1]$ with scalars in $\mathbb{C}$. For functions $u, v \in V$ and a scalar $\alpha \in \mathbb{C}$

$$u + v, \qquad \alpha u$$

are again functions in V.

We can define an **inner product** as follows: For functions $u(x), v(x)$ on $[0,1]$ let

$$(u,v) = \int_0^1 u(x)\overline{v(x)}\,dx.$$

Then we have the norm

$$\|u\|^2 = (u,u) = \int_0^1 |u(x)|^2\,dx.$$

We can also allow more general functions u as vectors (e.g. piecewise continuous), as long as $\int_0^1 |u(x)|^2\,dx$ is finite. The vector space of these functions (where the integrals exist in the sense of Lebesgue) is called $\boxed{L^2([0,1])}$, and the norm $\|u\|$ is called L^2-norm.

For functions $u^{(1)}, \ldots, u^{(N)}$ which are orthogonal on each other we consider again the subspace

$$W = \operatorname{span}\left\{u^{(1)}, \ldots, u^{(N)}\right\}.$$

For a given function v we want to find the best approximation $w = c_1 u^{(1)} + \cdots + c_N u^{(N)} \in V$, in the sense that the error

$$\|v-w\|^2 = \int_0^1 |v(x)-w(x)|^2\,dx$$

becomes minimal ("least squares approximation"). We can find v as follows:

1. Compute the coefficients $c_k = \frac{(w,u^{(k)})}{(u^{(k)},u^{(k)})}$ for $k = 1, \ldots, N$
2. Let $w = c_1 u^{(1)} + \cdots + c_N u^{(N)}$.

Example: Let $u^{(1)}(x) = 1$ and $u^{(2)}(x) = \begin{cases} 1 & x \in [0, \frac{1}{2}] \\ -1 & x \in (\frac{1}{2}, 1] \end{cases}$. Then the subspace $W := \operatorname{span}\{u^{(1)}, u^{(2)}\}$ consists of piecewise constant functions on the partition $0, \frac{1}{2}, 1$. Note that we have $\left(u^{(1)}, u^{(2)}\right) = 0$, i.e., we have an orthogonal basis.

For the given function $v(x) = x^2$ find the best least-squares approximation with a function $w \in W$:

We first compute the coefficients

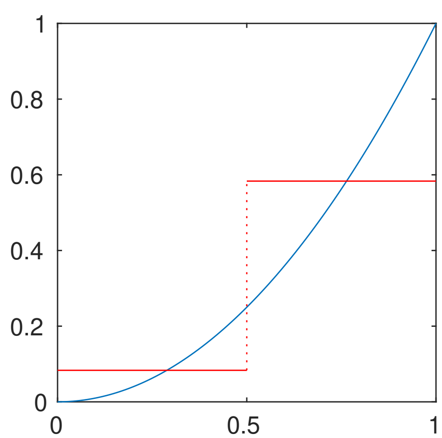
$$c_1 = \frac{(v,u^{(1)})}{(u^{(1)},u^{(1)})} = \frac{\int_0^1 x^2 dx}{\int_0^1 1 dx} = \frac{1/3}{1} = \frac{1}{3}, \qquad c_2 = \frac{(v,u^{(2)})}{(u^{(2)},u^{(2)})} = \frac{\int_0^{1/2} x^2 dx + \int_{1/2}^1 (-x^2) dx}{\int_0^1 1 dx} = \frac{\frac{1}{3}\left(\frac{1}{8} - 1 + \frac{1}{8}\right)}{1} = -\frac{1}{4}$$

and obtain

$$w(x) = c_1 u^{(1)}(x) + c_2 u^{(2)}(x) = \begin{cases} \frac{1}{12} & x \in [0, \frac{1}{2}] \\ \frac{7}{12} & x \in (\frac{1}{2}, 1] \end{cases}$$

We can compute $\|v-w\|$ using (8):

$$\|v-w\|^2 = \|v\|^2 - |c_1|^2 \left\|u^{(1)}\right\|^2 - |c_2|^2 \left\|u^{(2)}\right\|^2 = \tfrac{1}{5} - (\tfrac{1}{3})^2 \cdot 1 - (\tfrac{1}{4})^2 \cdot 1 = \frac{19}{720} = .02638888...$$



3 Periodic functions, continuous time: Fourier series

3.1 The space $\mathscr{T}_N$ of trigonometric functions

We are now interested in periodic functions: We say a function is periodic with period L if

$$f(x+L) = f(x) \qquad \text{for all } x \in \mathbb{R}.$$

We first consider the case of functions with period $L = 1$ (then the case with general period L will follow easily). We assume we have a "signal" $f(x)$ with

$$f(x+1) = f(x) \qquad \text{for all } x \in \mathbb{R}.$$

The function $f(x)$ may have complex values. We want to approximate this signal using the following trigonometric functions with period 1

$$\begin{aligned} v(x) = & a_0 \cdot 1 \\ & + a_1 \cos(2\pi x) + b_1 \sin(2\pi x) \\ & + \cdots \\ & + a_N \cos(2\pi N x) + b_N \cos(2\pi N x). \end{aligned} \tag{13}$$

with coefficients $a_k, b_k \in \mathbb{C}$. **We denote the space of these functions by $\mathscr{T}_N$ (trigonometric polynomials with frequency up to** N**)**. Note that $v(x)$ is a sum of terms with different frequencies:

- 1 has frequency 0
- $\cos(2\pi x)$ and $\sin(2\pi x)$ have frequency 1
- $\vdots$
- $\cos(2\pi N x)$ and $\sin(2\pi N x)$ have frequency N.

We consider the case of a complex valued signal, so we have $a_k, b_k \in \mathbb{C}$.

Recall that

$$e^{ix} = \cos x + i \sin x \tag{14}$$

implying

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}. \tag{15}$$

Using (15) in (13) gives the function $v(x)$ in the form

$$\begin{aligned} v(x) = & \hat{f}_0 \cdot 1 && \text{frequency } 0 \\ & + \hat{f}_{-1} e^{-2\pi i x} + \hat{f}_1 e^{2\pi i x} && \text{frequency } 1 \\ & \vdots && \vdots \\ & + \hat{f}_{-N} e^{-2\pi i N x} + \hat{f}_N e^{2\pi i N x}. && \text{frequency } N \end{aligned} \tag{16}$$

From this form of $v(x)$ we can get to (13) by using (14). The function $v(x)$ is written as a sum of terms with frequencies $0, 1, \ldots, N$.

For a real-valued function $f(x)$ the **coefficients** a_k, b_k **in (13) are real**, but the **coefficients** $\hat{f}_{-N}, \ldots, \hat{f}_N$ **in (16) are complex numbers** which need not be real.

We use the form (16) for finding $v(x)$. We can then easily convert this to the form (13). Therefore we use the functions

$$u^{(k)}(x) := e^{2\pi i k x} \qquad \text{for } k \in \mathbb{Z}$$

and have

$$\boxed{\mathscr{T}_N = \operatorname{span}\{u^{(-N)}, u^{(-N+1)}, \ldots, u^{(N)}\}}$$

Observations:

1. We have $u^{(k)}(x) \cdot u^{(\ell)}(x) = u^{(k+\ell)}(x)$. Therefore $u \in \mathscr{T}_N$ and $v \in \mathscr{T}_M$ implies for the product $u \cdot v \in \mathscr{T}_{N+M}$.
2. We have $\overline{u^{(l)}} = \cos(2\pi kx) - i\sin(2\pi ikx) = e^{-2\pi ilx}$ and hence

$$\left(u^{(k)}, u^{(l)}\right) = \int_0^1 e^{2\pi ikx} e^{-2\pi ilx} dx = \begin{cases} 0 & \text{for } k \neq l \\ 1 & \text{for } k = l \end{cases}$$

This means that the functions $u^{(-N)}, u^{(-N+1)}, \ldots, u^{(N)}$ form an **orthonormal basis** of the space $\mathscr{T}_N$.

For a given function $f(x)$ we denote the best approximation in the space $\mathscr{T}_N$ by $f_{(N)}(x)$:

$$f_{(N)}(x) = \sum_{k=-N}^{N} \hat{f}_k e^{2\pi ikx}$$

The coefficients $\hat{f}_k$ are called the **Fourier coefficients** of the function f. Since we have an orthonormal basis we can use (10) to compute them:

$$\hat{f}_k = \left(f, u^{(k)}\right) = \int_0^1 f(x) e^{-2\pi ikx} dx.$$

Since the functions are 1-periodic we can also use $\int_{-1/2}^{1/2} \cdots$ (or any other interval of length 1).

Proposition 1. *If the function f is real-valued, then $\hat{f}_{-k} = \overline{\hat{f}_k}$ and the function $f_{(N)}$ is real-valued.*

Proof. We have for the Fourier coefficients $\hat{f}_k$ and its complex conjugate $\overline{\hat{f}_k}$

$$\hat{f}_k = \int_0^1 f(x) e^{-2\pi ikx} dx, \qquad \overline{\hat{f}_k} = \int_0^1 \overline{f(x)}\, \overline{e^{-2\pi ikx}} dx = \int_0^1 f(x) e^{2\pi ikx} = \hat{f}_{-k}$$

Hence we get for the complex conjugate of $f_{(N)}(x) = \sum_{k=-N}^{N} \hat{f}_k e^{2\pi ikx}$

$$\overline{f_{(N)}(x)} = \sum_{k=-N}^{N} \overline{\hat{f}_k}\, \overline{e^{2\pi ikx}} = \sum_{k=-N}^{N} \hat{f}_{-k} e^{2\pi i(-k)x} = f_{(N)}(x)$$

since this is the same sum, just in reverse order. □

Note that this argument works since we use a symmetric range $k = -N, \ldots, N$ of indices. For a real-valued function $f(x)$ the function $g(x) := \sum_{k=0}^{N} \hat{f}_k e^{2\pi ikx}$ is in general non-real.

For a general complex valued function $f(x)$ we can write the terms of frequency k as

$$\begin{aligned} \hat{f}_{-k} u^{(-k)} + \hat{f}_k u^{(k)} &= \hat{f}_{-k} [\cos(2\pi kx) - i\sin(2\pi kx)] + \hat{f}_k [\cos(2\pi kx) + i\sin(2\pi kx)] \\ &= \left(\hat{f}_k + \hat{f}_{-k}\right) \cos(2\pi kx) + i\left(\hat{f}_k - \hat{f}_{-k}\right) \sin(2\pi kx) \end{aligned} \tag{17}$$

For a real valued function $f(x)$ we get complex Fourier coefficients $\boxed{\hat{f}_k = A_k + iB_k}$ and $\boxed{\hat{f}_{-k} = A_k - iB_k}$ with real A_k, B_k yielding

$$\hat{f}_{-k} u^{(-k)} + \hat{f}_k u^{(k)} = \boxed{2A_k \cos(2\pi kx) - 2B_k \sin(2\pi kx)} \tag{18}$$

The equation (11) is here

$$\|f\|^2 = \left|\hat{f}_{-N}\right|^2 + \cdots + \left|\hat{f}_N\right|^2 + \left\|f - f_{(N)}\right\|^2 \tag{19}$$

which can be used to find $\left\|f - f_{(N)}\right\|$. Because of $\left\|f - f_{(N)}\right\| \geq 0$ we have

$$\|f\|^2 \geq \sum_{k=-N}^{N} \left|\hat{f}_k\right|^2.$$

Since this is true for any $N = 0, 1, 2, 3, \ldots$ we have the **Bessel inequality**

$$\|f\|^2 \geq \sum_{k=-\infty}^{\infty} \left|\hat{f}_k\right|^2 \tag{20}$$

where the sum $S := \sum_{k=-\infty}^{\infty} \left|\hat{f}_k\right|^2 \leq \|f\|^2$ converges since it is bounded from above. Hence we have

$$\left\|f - f_{(N)}\right\| \to \|f\|^2 - S \quad \text{as } N \to \infty. \tag{21}$$

We have $S \leq \|f\|^2$. Is $S = \|f\|^2$? If yes, the Fourier series $f_{(N)}$ converges to f as $N \to \infty$.

As $\sum_{k=-\infty}^{\infty} \left|\hat{f}_k\right|^2$ converges we must have

$$\boxed{\hat{f}_k \to 0 \quad \text{as } k \to \infty \text{ or } k \to -\infty} \tag{22}$$

i.e., the Fourier coefficients decay to zero as the frequency $|k|$ increases.

3.2 Example: Fourier series and convergence

Consider the 1-periodic function $f(x)$ with

$$f(x) = \begin{cases} -1 & \text{for } x \in [-\frac{1}{2}, 0) \\ 1 & \text{for } x \in [0, \frac{1}{2}) \end{cases} \tag{23}$$

and find the best approximation $f_{(5)}(x)$ in the space $\mathscr{T}_5$.

We need to compute the Fourier coefficients $\hat{f}_k = \left(f, u^{(k)}\right)$. We start with $\hat{f}_0$: Since $u^{(0)}(x) = 1$ we get

$$\hat{f}_0 = \int_{-1/2}^{1/2} f(x) \cdot 1 dx = \int_{-1/2}^{0} (-1) dx + \int_{0}^{1/2} 1 dx = -\tfrac{1}{2} + \tfrac{1}{2} = 0.$$

Now we consider $k \neq 0$:

$$\hat{f}_k = \int_{-1/2}^{1/2} f(x) e^{-2\pi i k x} dx = \int_{-1/2}^{0} (-1) e^{-2\pi i k x} dx + \int_{0}^{1/2} 1 \cdot e^{-2\pi i k x} dx \tag{24}$$

For the second integral we get

$$\int_{0}^{1/2} e^{-2\pi i k x} dx = \left[\frac{e^{-2\pi i k x}}{-2\pi i k}\right]_{x=0}^{1/2} = \frac{-1}{2\pi i k}\left(e^{-2\pi i k/2} - 1\right)$$

We have for integer k that

$$e^{-2\pi i k/2} = \begin{cases} 1 & \text{for even } k \\ -1 & \text{for odd } k \end{cases}$$

hence

$$\int_{0}^{1/2} e^{-2\pi i k x} dx = \begin{cases} 0 & \text{for even } k \\ \frac{-1}{2\pi i k}(-2) = -\frac{i}{\pi k} & \text{for odd } k \end{cases}$$

For the first integral in (24) we proceed in the same way and get exactly the same value. Therefore we obtain the Fourier coefficients

$$\hat{f}_k = \begin{cases} 0 & \text{for even } k \in \mathbb{Z} \\ -\frac{2i}{\pi k} & \text{for odd } k \in \mathbb{Z} \end{cases} \tag{25}$$

and the best approximation $f_{(5)} \in \mathscr{T}_5$ is

$$f_{(5)}(x) = -\frac{2i}{\pi}\left(-\frac{1}{5}e^{-10\pi i x} - \frac{1}{3}e^{-6\pi i x} + e^{-2\pi i x} + e^{2\pi i x} + \frac{1}{3}e^{6\pi i x} + \frac{1}{5}e^{10\pi i x}\right).$$

Note that we have terms with frequency 1, 3 and 5. We now want to convert this to the form (13): E.g., for the terms of frequency 3 we get with (14)

$$-\frac{2i}{\pi}\left(-\frac{1}{3}e^{-6\pi ix}+\frac{1}{3}e^{6\pi ix}\right)=-\frac{2i}{\pi}\cdot\left(\frac{i}{3}\cdot\sin(6\pi x)+\frac{i}{3}\sin(6\pi x)\right)=\frac{4}{3\pi}\sin(6\pi x)$$

since the cosine terms cancel. Therefore we get

$$f_{(5)}(x)=\frac{4}{\pi}\left(\sin(2\pi x)+\frac{1}{3}\sin(3\cdot 2\pi x)+\frac{1}{5}\sin(5\cdot 2\pi x)\right)$$

For a general odd N we have

$$f_{(N)}(x)=\frac{4}{\pi}\left(\frac{1}{1}\sin(1\cdot 2\pi x)+\frac{1}{3}\sin(3\cdot 2\pi x)+\frac{1}{5}\sin(5\cdot 2\pi x)+\cdots+\frac{1}{N}\sin(N\cdot 2\pi x)\right)$$

In order to evaluate a Fourier sum $\sum_{k=-N}^{N}\hat{f}_k e^{2\pi ikx}$ in Matlab we use **`foursum(x,fh0,fhpos,fhneg)`**. Here

$$\texttt{fh0}=\hat{f}_0,\qquad \texttt{fhpos}=\left[\hat{f}_1,\ldots,\hat{f}_N\right],\qquad \texttt{fhneg}=\left[\hat{f}_{-1},\ldots,\hat{f}_{-N}\right]$$

For a real-valued function we can omit the last argument and use **`foursum(x,fh0,fhpos)`**.

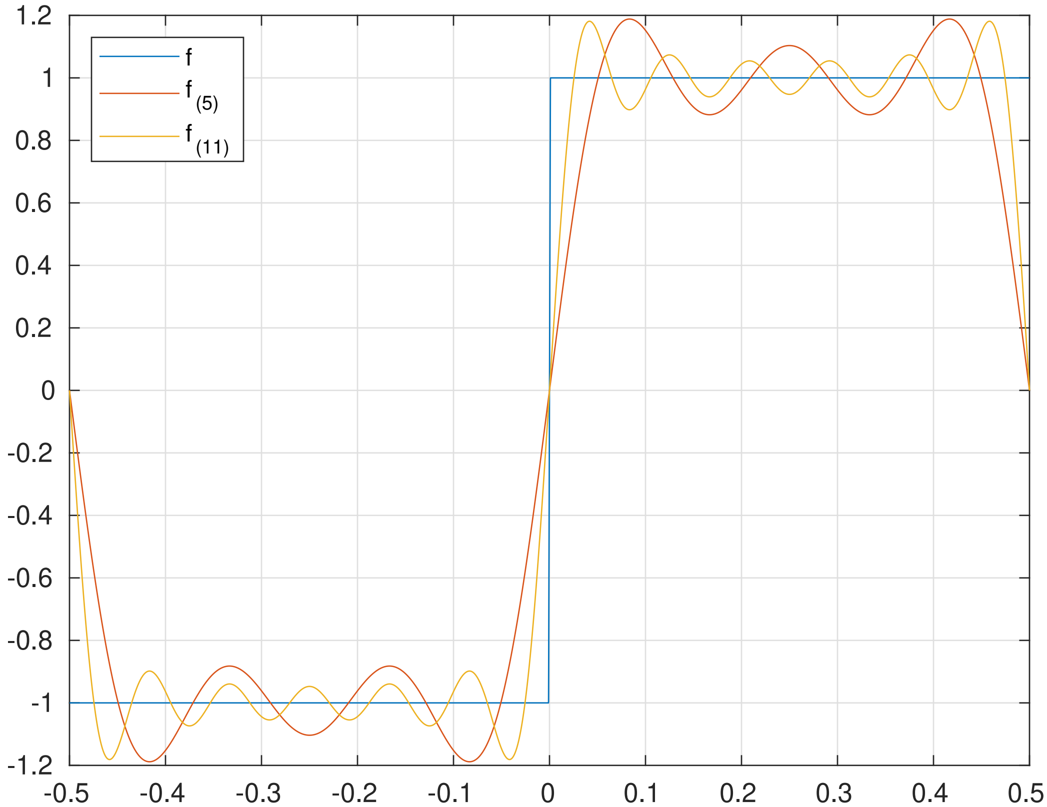
Here is the m-file **`foursum.m`**:

```
function y = foursum(x,fh0,fhpos,fhneg)
if nargin==3                                % foursum(x,fh0,fhpos) uses fhneg=conj(fhpos)
 fhneg = conj(fhpos);
end

f = fh0*ones(size(x));
for k=1:length(fhpos)
  f = f + fhpos(k)*exp(k*2i*pi*x);
end
for k=1:length(fhneg)
  f = f + fhneg(k)*exp(-k*2i*pi*x);
end
```

We can then plot the functions $f(x)$, $f_{(5)}(x)$ and $f_{(11)}(x)$ on the interval $[-\frac{1}{2},\frac{1}{2}]$ together:

```
x = -.5:.001:.5;                            % points for plotting
fhpos = -2i/pi./(1:11); fhpos(2:2:11)=0;    % Fourier coefficients for even k are zero
plot(x,sign(x),x,foursum(x,0,fhpos(1:5)),x,foursum(x,0,fhpos))
legend('f','f_{(5)}','f_{(11)}'); grid on
```



Observations:

- For $x = 0$ we have $f(0) = 1$ according to the definition of f, but $f_{(N)}(x) = 0$ for all N. This is not surprising: If we change the definition of $f(0)$ e.g. from 1 to -1 the integrals $\hat{f}_k = \left(f, u^{(k)}\right)$ do not change. Therefore changing the definition of f in a single point does not affect $f_{(N)}(x)$.
- There is an "**overshoot**" near the jump:
  The smallest local maximum has a y-value of about 1.1789, for any value of N.
  For each N there exists $x \in (0, \frac{1}{2})$ with $f_{(N)}(x) \geq 1.1789$. As N increases, the "overshoot" happens closer and closer to the jumps, and away from the jump the function $f_{(N)}$ gets closer and closer to f.
  Since the exact function value is 1 the size of the overshoot is ≈ 0.1789. Note that the jump size at $x = 0$ is $J = 2$.

  The size of the overshoot is always $\approx .0895J$ where J denotes the jump size

  We will later see that the number .0895 is actually obtained in terms of the "sine integral" function:

```
>> overshoot = sinint(pi)/pi-.5
overshoot =
        0.0894898722360836
```

We now want to find the error $\left\|f - f_{(N)}\right\|$ by using (19). Here we have

$$\|f\|^2 = \int_{-1/2}^{1/2} |f(x)|^2 \, dx = \int_{-1/2}^{1/2} 1 dx = 1$$

and for odd N

$$\sum_{k=-N}^{N} |\hat{f}_k|^2 = \left(\frac{2}{\pi}\right)^2 2 \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{N^2}\right).$$

Hence

$$\|f - f_{(5)}\| = \sqrt{1 - \frac{8}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\right)} \approx 0.25874$$

$$\|f - f_{(11)}\| = \sqrt{1 - \frac{8}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{11^2}\right)} \approx 0.18357$$

What will happen as N tends to infinity? One can show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \pi^2/8$. The Matlab symbolic toolbox can find this symbolic sum:

```
>> syms k; symsum(1/(2*k-1)^2,k,1,Inf)
ans =
pi^2/8
```

Hence

$$S = \sum_{k=-\infty}^{\infty} \left|\hat{f}_k\right|^2 = \frac{8}{\pi^2} \underbrace{\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)}_{\pi^2/8} = 1$$

so we get from (21)

$$\|f - f_{(N)}\| \to \|f\|^2 - S = 1 - 1 = 0 \quad \text{as } N \to \infty$$

which means that the Fourier series

$$\sum_{k=-\infty}^{\infty} \hat{f}_k e^{2\pi i k x} = \frac{4}{\pi} \sum_{\substack{k=1\ldots\infty \\ k \text{ odd}}} \frac{\sin(k \cdot 2\pi x)}{k} \tag{26}$$

converges to the function f in the mean square sense.

This does not mean that we have "pointwise convergence": We have $f(0) = 1$, but $f_{(N)}(0) = 0$.

If we modify the function f at finitely many points (or even countably many points), the new function $\tilde{f}$

- satisfies $\|\tilde{f} - f\| = 0$, i.e., these two functions are considered to be the same vector in L^2.
- $\tilde{f}$ has the same Fourier coefficients as f

3.3 The three basic convergence theorems for Fourier series

We defined the Fourier approximation $f_{(N)}(x) = \sum_{k=-N}^{N} \hat{f}_k e^{2\pi i k x}$. We want to let $N \to \infty$ and obtain the **Fourier series**

$$\sum_{k=-\infty}^{\infty} \hat{f}_k e^{2\pi i k x}$$

But this expression only makes sense as a limit $N \to \infty$. In which sense does this limit exist?

It turns out that there are three different ways to understand this limit. This is the key to understanding the concept of a "Fourier series".

$\boxed{L^2}$ **is the space of 1-periodic functions** $f(x)$ **where** $\int_0^1 |f(x)|^2 \, dx$ **exists** (as a Lebesgue integral). Recall that functions $f, \tilde{f}$ who differ at finitely many points (or countably many points, or on a set of measure zero) have $\|f - \tilde{f}\| = 0$ and are therefore considered to be the same vector in L^2.

Let $\boxed{PW^0}$ **denote the space of piecewise continuous 1-periodic functions**: f is continuous at all $z \in [0, 1)$ except for finitely many breakpoints $z_1, \ldots, z_M \in [0, 1)$, let $z_0 := z_M$. At each breakpoint z_j the left-sided limit $f(z_j - 0)$ and the right-sided limit $f(z_j + 0)$ need to exist. Note that functions in PW^0 **may have jumps**.

Let us denote the "pieces with limits" on $[z_{j-1}, z_j]$ by $f_j(x)$:

$$f_j(x) := \begin{cases} f(z_{j-1} + 0) & x = z_{j-1} \\ f(x) & z_{j-1} < x < z_j \\ f(z_j - 0) & x = z_j \end{cases}$$

Let $\boxed{PW^1}$ **denote the space of piecewise continuously differentiable 1-periodic functions:** Assume $f \in PW^0$ and that each piece $f_j(x)$ has a continuous derivative $f_j'(x)$ on $[z_{j-1}, z_j]$. Note that functions in PW^1 **may have jumps**.

We will later prove the following three theorems:

(T1) For all $\boxed{f \in L^2}$ we have

$$\boxed{\left\| f_{(N)} - f \right\| \to 0 \quad \text{as } N \to \infty} \qquad \textbf{mean square convergence} \tag{27}$$

(T2) For all $\boxed{f \in PW^1}$ we have for all $x \in \mathbb{R}$:

$$\boxed{f_{(N)}(x) \to \frac{f(x-0)+f(x+0)}{2} \quad \text{as } N \to \infty} \qquad \textbf{pointwise convergence} \tag{28}$$

I.e., if f does not have a jump at x_0 we have $f_{(N)}(x_0) \to f(x_0)$. If f has a jump at x_0 the Fourier series converges to the average of left and right-sided limit, irrespective of how $f(x_0)$ is defined.

(T3) For all $\boxed{f \in PW^1 \text{ without jumps}}$ we have

$$\boxed{\max_{x \in [0,1)} \left| f_{(N)}(x) - f(x) \right| \to 0 \quad \text{as } N \to \infty} \qquad \textbf{uniform convergence} \tag{29}$$

In this case we also have that $\sum_{k=-\infty}^{\infty} \left| \hat{f}_k \right| < \infty$, i.e. we have for each $x \in \mathbb{R}$ "absolute convergence" of the Fourier series $\sum_{-\infty}^{\infty} \hat{f}_k e^{2\pi i k x}$.

Note that uniform convergence implies pointwise convergence and mean square convergence.

Example 1: Consider the 1-periodic function $f(x) = x^{-1/2}$ for $x \in [0,1)$. We can define the Fourier coefficients $\hat{f}_k = \int_0^1 x^{-1/2} e^{-2\pi i k x} dx$. These integrals exist since $\left| x^{-1/2} e^{-2\pi i k x} \right| = x^{-1/2}$ which is integrable on $[0,1]$.

But we have $\int_0^1 |f(x)|^2 dx = \int_0^1 x^{-1} dx$, and this integral does not exist as $\int_\varepsilon^1 x^{-1} dx = \ln 1 - \ln \varepsilon \to \infty$ as $\varepsilon \to 0$. Hence $f \notin L^2$, and we do not have mean square convergence.

The Dini criterion (30) gives pointwise convergence $f_{(N)}(x) \to f(x)$ for all $x \in (0,1)$, but not for $x = 0$.

Example 2: Consider the 1-periodic function $f(x) = x^{-1/3}$ for $x \in [0,1)$. Here we have $\int_0^1 |f(x)|^2 dx = \int_0^1 x^{-2/3} dx = \left[3x^{1/3}\right]_0^1 = 3$, so $f \in L^2$ and we have mean square convergence $\left\| f_{(N)} - f \right\| \to 0$ as $N \to \infty$.

The Dini criterion (30) gives pointwise convergence $f_{(N)}(x) \to f(x)$ for all $x \in (0,1)$, but not for $x = 0$.

Example 3: Consider the 1-periodic function $f(x) = x^2$ for $x \in [0,1]$. Clearly $f \in PW^1$, and therefore we have pointwise convergence. For $x = 0$ we have $f_{(N)}(x) \to \frac{1}{2}$ as $N \to \infty$.

Example 4: Let f denote the 1-periodic function with $f(x) = |x|$ for $x \in [-\frac{1}{2}, \frac{1}{2})$. Clearly $f \in PW^1$, and f does not have jumps. Hence by (T3) we have uniform convergence.

Remarks:

1. **Continuity is not enough to guarantee pointwise convergence everywhere** :
   There exists a continuous function f such that $\sup_{N \in \mathbb{N}} f_{(N)}(0) = \infty$. There are even functions where $\sup_{N \in \mathbb{N}} f_{(N)}(x) = \infty$ for all x in an uncountable dense subset of $[0,1]$.
   However: If f is continuous or even $f \in L^2$ there is pointwise convergence $f_{(N)}(x) \to f(x)$ for all $x \in [0,1]$ except on a set of measure zero.

2. For $f \in PW^1$ with jumps we have $\left| \hat{f}_k \right| = O(|k|^{-1})$ and therefore no absolute convergence.
   Absolute convergence implies uniform convergence. However, there are Fourier series with uniform convergence, but no absolute convergence.

3. For (T2), (T3) one can weaken the assumption $f \in PW^1$. We don't need that each piece f_j has a continuous derivative. Actually it is enough that **each piece $f_j(x)$ is "slightly better than continuous"**:
   If $\left| f_j(s) - f_j(t) \right| \le C |s-t|^\alpha$ with some $\alpha > 0$ ("Hölder continuity") then (28) still holds.
   If each f_j is additionally piecewise monotonic and there are no jumps then (29) and absolute convergence still holds.

4. The fundamental result for pointwise convergence is the "**Dini criterion**": If f is integrable and

$$\int_0^{1/2} \left| \frac{f(x_0-t)+f(x_0+t)}{2} - L \right| \frac{dt}{t} < \infty \tag{30}$$

we have $f_{(N)}(x_0) \to L$ as $N \to \infty$.

MORAL: There are some evil artificial continuous functions for which pointwise convergence fails in many points. But for all practical examples from applications f is piecewise "slightly better than continuous", and then we have **pointwise convergence everywhere** (28), and in the absence of jumps **uniform convergence** (29).

3.4 Convolutions

Convolutions can be used for signal processing. We can e.g. "smoothen" a signal using convolutions.

For 1-periodic functions f, g we define the **convolution** $q = f * g$ by

$$\boxed{q(x) = \int_{t=0}^{1} f(t)g(x-t)dt} \tag{31}$$

Properties of the convolution:

- q is also 1-periodic
- $f * g = g * f$ since using the change of variable $s = x - t$ in (31) gives $t = x - s$, $dt = -ds$

$$q(x) = \int_{s=x}^{x-1} f(x-s)g(s)(-ds) = \int_{s=x-1}^{x} f(x-s)g(s)ds = \int_{s=0}^{1} f(x-s)g(s)ds$$

  (as the integrand is 1-periodic in s, we have $\int_{s=x-1}^{x}(\cdots)ds = \int_{s=0}^{1}(\cdots)ds$.)
- $(f*g)' = f' * g = f * g'$: Assume that f'exists and is continuous, g is continuous. Then we can take the derivative $\frac{d}{dx}$ in (31) inside the integral:

$$q'(x) = \int_{t=0}^{1} f(t)g'(x-t)dt = f * g'.$$

- If $g \in \mathscr{T}_n$ then also $q = f * g \in \mathscr{T}_n$: For $g(x) = \sum_{k=-N}^{N} \hat{g}_k e^{2\pi i k x}$ we get

$$q(x) = \int_{t=0}^{1} f(t) \left[\sum_{k=-N}^{N} \hat{g}_k e^{2\pi i k(x-t)} \right] dt = \sum_{k=-N}^{N} \underbrace{\left(\int_{t=0}^{1} f(t) e^{-2\pi i k t} dt \right)}_{\hat{f}_k} \hat{g}_k e^{2\pi i k x} = \sum_{k=-N}^{N} \hat{q}_k e^{2\pi i k x} \tag{32}$$

  with

$$\boxed{\hat{q}_k = \hat{f}_k \hat{g}_k} \tag{33}$$

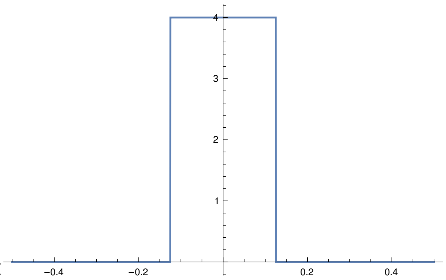
A simple example of a convolution is smoothing with a "moving average": Define the 1-periodic function g_n for $x \in [-\frac{1}{2}, \frac{1}{2})$ by

$$g_n(x) = \begin{cases} n & \text{for } x \in (-\frac{1}{2n}, \frac{1}{2n}) \\ 0 & \text{otherwise} \end{cases} \tag{34}$$

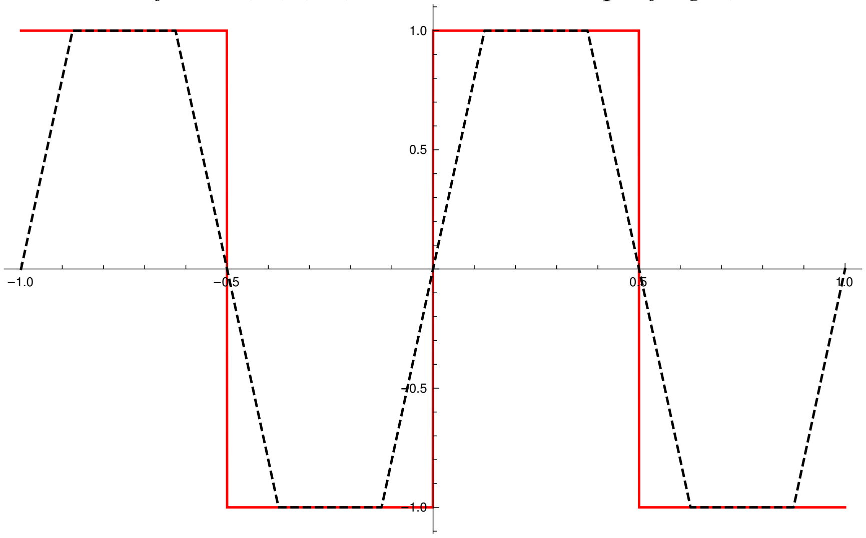
Note that $\int_{-1/2}^{1/2} g_n(x)dx = 1$. For a 1-periodic function f the convolution $q_n = f * g_n$ gives

$$q_n(x) = n \int_{t=x-1/(2n)}^{x+1/(2n)} f(t)dt$$

which is the average of the function f over the interval $[x - \frac{1}{2n}, x + \frac{1}{2n}]$.

Example: Here is the function $g_4(x)$: 

We consider f from (23) (red) and the convolution $q = f * g_4$ (black, dashed):



It is easy to see that we must have $q(x) = 1$ for $x \in [\frac{1}{8}, \frac{3}{8}]$ since we only take the average of function values 1. Similarly, $q(x) = -1$ for $x \in [-\frac{3}{8}, -\frac{1}{8}]$ since we only take the average of function values -1. For $x \in [-\frac{1}{8}, \frac{1}{8}]$ the value of $q(x)$ increases from -1 to 1, with $q(0) = 0$. One can show: the convolution of two piecewise constant functions f, g gives a continuous piecewise linear function $q = f * g$.

If the function f is continuous the maximal error will converge to 0 ("**uniform convergence**"):

$$\max_{x\in\mathbb{R}} |f(x) - q_n(x)| \to 0 \quad \text{as } n \to \infty \tag{35}$$

Proof: For any $\varepsilon > 0$ there exists $\delta > 0$ so that $|y - x| < \delta$ implies $|f(y) - f(x)| < \varepsilon$. Hence we have for $n \geq 1/\delta$ that $|f(t) - f(x)| < \varepsilon$ for all $t \in [x - \frac{1}{2n}, x + \frac{1}{2n}]$. Then $q_n(x) - f(x) = n \int_{t=x-1/(2n)}^{x+1/(2n)} (f(t) - f(x))\, dt$ gives

$$|q_n(x) - f(x)| \leq n \int_{t=x-1/(2n)}^{x+1/(2n)} \underbrace{|f(t) - f(x)|}_{\leq \varepsilon} \leq \varepsilon \qquad \square$$

If $\max |f - q_n| \leq \varepsilon$ we have $\|f - q_n\|^2 \leq \int_0^1 \varepsilon^2 dx = \varepsilon^2$. I.e., uniform convergence implies $\|f - q_n\| \to 0$ (convergence in mean square sense).

The functions g_n become "more and more concentrated near 0**"** in the following sense: We have for $n = 1, 2, 3, \ldots$

(P1) $g_n(x) \geq 0$

(P2) $\int_{-1/2}^{1/2} g_n(x) dx = 1$

(P3) for every $\delta > 0$: $\max_{[-\frac{1}{2}, \frac{1}{2}] \setminus [-\delta, \delta]} g_n(x) \to 0$ as $n \to \infty$

For any sequence of functions g_n with these three properties we can prove uniform convergence of $f * g_n$:

Lemma 2. *Assume that the 1-periodic functions $g_1, g_2, g_3, \ldots$ satisfy the properties (P1), (P2), (P3). For a 1-periodic continuous function f we have for $q_n := f * g_n$*

$$\max_{x\in\mathbb{R}} |f(x) - q_n(x)| \to 0 \quad \textit{as } n \to \infty$$

Proof. For a given $\varepsilon > 0$ we want to show $\max_{x\in\mathbb{R}} |f(x) - q_n(x)| < \varepsilon$ if n is sufficiently large. Since f is continuous on $[-\frac{1}{2}, \frac{1}{2}]$ there exists $\delta > 0$ such that

$$\text{for all } x,y \text{ with } |x-y| < \delta: \qquad |f(x) - f(y)| < \varepsilon/2 \tag{36}$$

With (P2) we obtain

$$\begin{aligned} q_n(x) - f(x) &= \int_{-1/2}^{1/2} [f(x-t) - f(x)]\, g_n(t)dt \\ &= \underbrace{\int_{[-\frac{1}{2},\frac{1}{2}]\setminus[-\delta,\delta]} [f(x-t) - f(x)]\, g_n(t)dt}_{①} + \underbrace{\int_{-\delta}^{\delta} [f(x-t) - f(x)]\, g_n(t)dt}_{②} \end{aligned}$$

Because of (36) we have

$$② < \int_{-\delta}^{\delta} \frac{\varepsilon}{2} g_n(t)dt \le \int_{-1/2}^{1/2} \frac{\varepsilon}{2} g_n(t)dt \stackrel{(P2)}{=} \frac{\varepsilon}{2}$$

Let $M = \max_{[-1/2,1/2]} |f(x)|$. Then we can use (P3) to find N such that

$$\text{for } n \ge N: \qquad \max_{[-\frac{1}{2},\frac{1}{2}]\setminus[-\delta,\delta]} g_n(x) \le \frac{\varepsilon}{4M}$$

which gives with $|f(x-t) - f(x)| \le 2M$ that

$$① \le \int_{[-\frac{1}{2},\frac{1}{2}]\setminus[-\delta,\delta]} 2M \frac{\varepsilon}{4M} dt \le \int_{[-\frac{1}{2},\frac{1}{2}]} 2M \frac{\varepsilon}{4M} dt = \frac{\varepsilon}{2}$$

□

The "limit" of g_n for $n \to \infty$ is something which is nonzero only for $x = 0$, but still has integral 1. This object is called **Dirac delta**. It is not a function, but a so-called *generalized function*. This means that pointwise evaluation $\delta(x)$ does not really make sense. However, $\int f(x)\delta(x)dx$ still makes sense and gives $f(0)$. See section 3.14 below for more details.

Here we consider the **1-periodic Dirac delta** δ_{per}. This can be imagined as the "limit" of the 1-periodic functions $g_n(x)$ as $n \to \infty$, so that we *formally* have $\int_{-1/2}^{1/2} F(x)\delta_{\text{per}}(x)dx = F(0)$ for any continuous function F. Therefore we can formally find the Fourier coefficients of $g = \delta_{\text{per}}$:

$$\hat{g}_k = \int_{-1/2}^{1/2} \delta_{\text{per}}(x) \underbrace{e^{-2\pi ikx}}_{F(x)} dx = F(0) = e^{-2\pi i\cdot 0} = 1 \qquad \text{for all } k.$$

We *formally* have $q = f * \delta_{\text{per}}$ with

$$q(x) = \int_{t=-1/2}^{1/2} \delta_{\text{per}}(t) \underbrace{f(x-t)}_{F(t)} dt = F(0) = f(x)$$

So for $g = \delta_{\text{per}}$ we *formally* have $q := f * g = f$, and for the Fourier coefficients we have

$$\hat{q}_k = \hat{f}_k \underbrace{\hat{g}_k}_{1} = \hat{f}_k$$

3.5 The Fourier approximation $f_{(N)}$ as convolution $f_{(N)} = f * D_N$ with the Dirichlet kernel D_N

We define the **Dirichlet kernel** $D_N \in \mathscr{T}_N$ by

$$D_N(x) = \sum_{k=-N}^{N} 1 \cdot e^{2\pi ikx}$$

Therefore (32), (33) give for the convolution $q = f * D_N$

$$q(x) = \sum_{k=-N} \hat{f}_k e^{2\pi ikx} = f_{(N)}(x)$$

Hence the Fourier approximation $f_{(N)}$ is the convolution with the Dirichlet kernel: $\boxed{f_{(N)} = f * D_N}$

The Dirichlet kernel is a 1-periodic real function. What does the graph look like?

We claim

$$D_N(x) := \sum_{k=-N}^{N} u^{(k)} = \begin{cases} 2N+1 & \text{for } x \in \mathbb{Z} \\ \dfrac{\sin((2N+1)\pi x)}{\sin(\pi x)} & \text{otherwise} \end{cases} \tag{37}$$

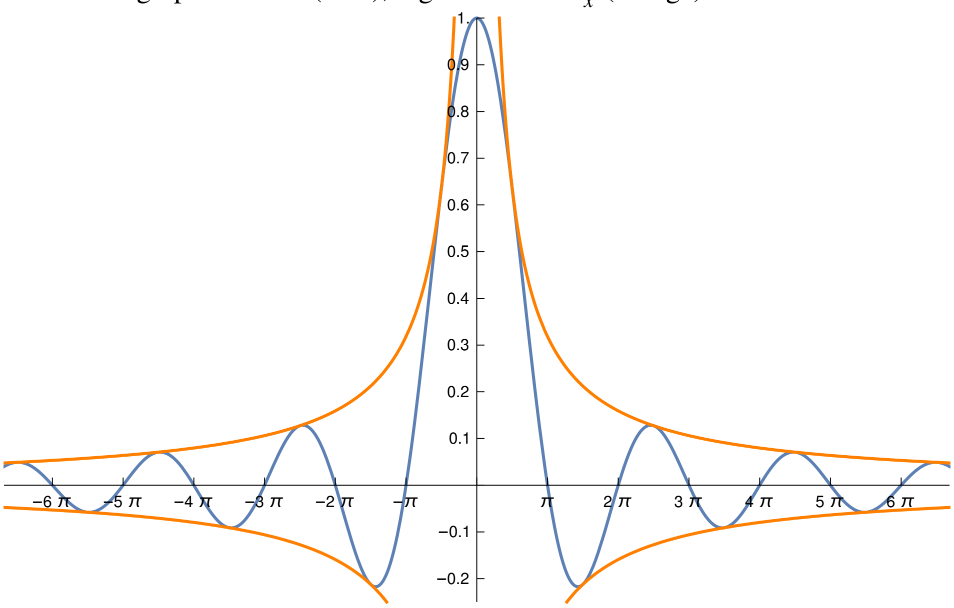
Proof: For $x = 0$ we obtain $D_N(0) = 2N+1$. For nonzero $x \in [-\frac{1}{2}, \frac{1}{2}]$ let $r := e^{2\pi ix}$

$$D_N(x) = r^{-N} + \cdots + r^N = r^{-N}\left(1 + r + \cdots + r^{2N}\right) = r^{-N}\frac{r^{2N+1}-1}{r-1} = \frac{r^{N+1}-r^{-N}}{r-1} = \frac{r^{N+\frac{1}{2}} - r^{-N-\frac{1}{2}}}{r^{1/2} - r^{-1/2}} \quad \square$$

The function $\dfrac{\sin x}{x}$ is defined for all $x \neq 0$. For $x = 0$ we have the limit 1. Therefore we define

$$\boxed{\operatorname{sinc} x := \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}}$$

Here is the graph of $\operatorname{sinc} x$ (blue), together with $\pm\frac{1}{x}$ (orange):



Warning: The function `sinc(x)` in Matlab is $\operatorname{sinc}_{\text{Matlab}}(x) = \operatorname{sinc}(\pi x)$.

We rewrite (37) using sinc and obtain: $D_N(x)$ is a 1-periodic function which for $x \in [-\frac{1}{2}, \frac{1}{2}]$ is given by

$$D_N(x) = (2N+1)\frac{\operatorname{sinc}((2N+1)\pi x)}{\operatorname{sinc}(\pi x)} \tag{38}$$

For $x \neq 0$

$$D_N(x) = \frac{1}{\pi x \operatorname{sinc}(\pi x)} \cdot \sin((2N+1)\pi x)$$

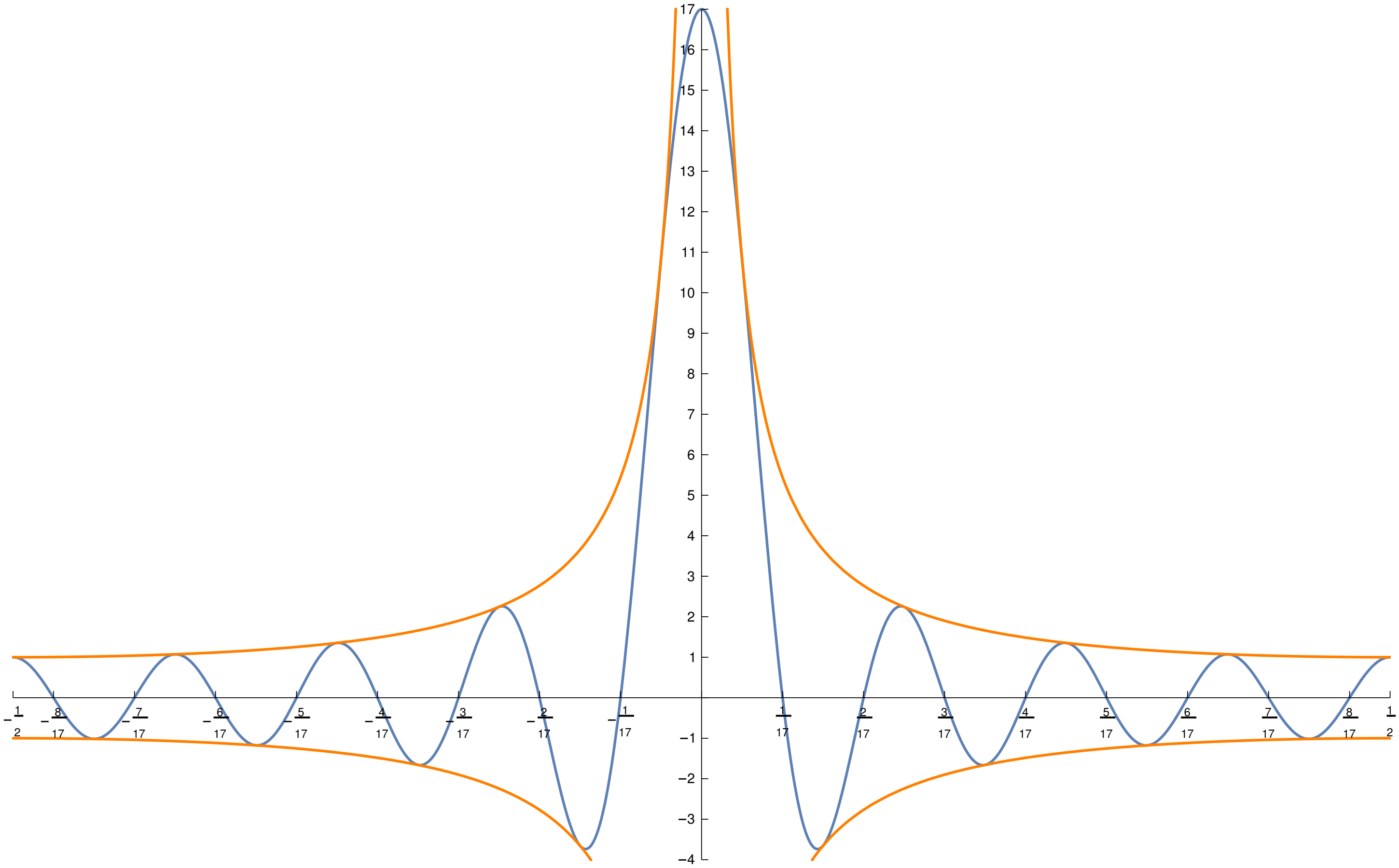
Note that $\operatorname{sinc}(\pi x)$ is an even function which decreases from 1 to $2/\pi$ on the interval $[0, \frac{1}{2}]$. Hence

$$G(x) := \frac{1}{\pi |x| \operatorname{sinc}(\pi x)} \tag{39}$$

satisfies on $[-\frac{1}{2}, \frac{1}{2}]$

$$\frac{1}{\pi |x|} \leq G(x) \leq \frac{1}{2|x|}$$

Here is the graph of $D_8(x)$ (blue) together with $\pm G(x)$ (orange, **independent of** N):

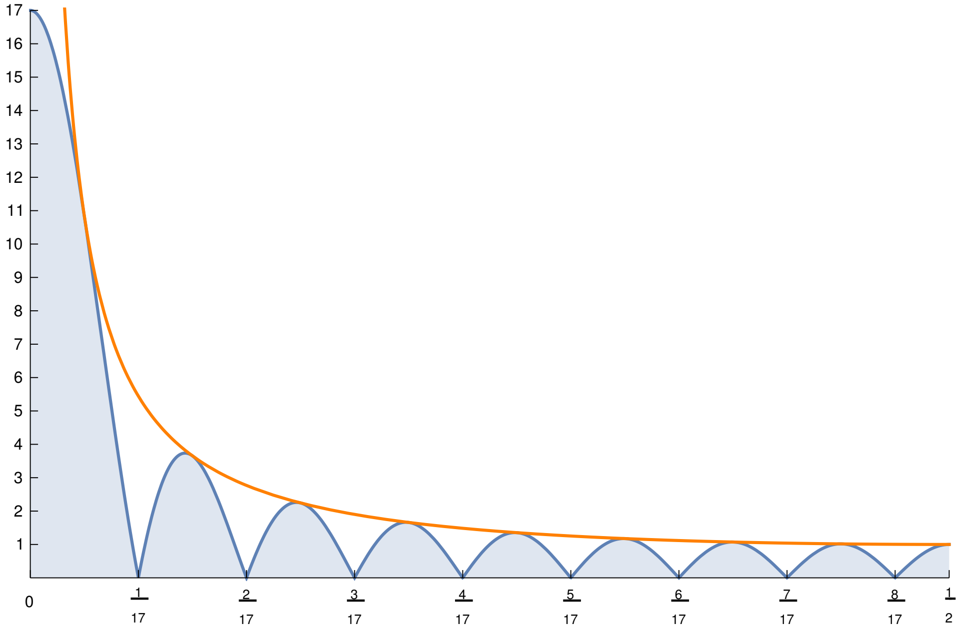


Since the function D_N has negative values it does not satisfy (P1).

But it also does not satisfy (P3) since the orange curves are independent of N.

Note that the upper orange curves behave like $\frac{c}{|x|}$ which is not integrable on $[-\frac{1}{2}, \frac{1}{2}]$. We will now show that because of this we have $\int_{-1/2}^{1/2} |D_N(x)| \, dx \to \infty$ as $N \to \infty$:

Consider $\int_0^{1/2} |D_N(x)| \, dx$: We consider the N "humps" at $[\frac{j}{2N+1}, \frac{j+1}{2N+1}]$ for $j = 0, \ldots, N-1$: here $N = 8$



We have

$$|D_N(x)| \geq \frac{|\sin((2N+1)\pi x)|}{\pi |x|}$$

Let $a_j := \int_{j/(2N+1)}^{(j+1)/(2N+1)} D_N(x)dx$. Then the area of the hump on $[\frac{j}{2N+1}, \frac{j+1}{2N+1}]$ is for $j = 0, \ldots, N-1$

$$|a_j| = \int_{j/(2N+1)}^{(j+1)/(2N+1)} |D_N(x)|\, dx \geq \frac{1}{\pi \frac{j+1}{2N+1}} \underbrace{\int_{j/(2N+1)}^{(j+1)/(2N+1)} |\sin((2N+1)\pi x)|\, dx}_{\frac{1}{2N+1} \underbrace{\int_0^1 \sin(\pi x) dx}_{2/\pi}} = \frac{2}{\pi^2} \cdot \frac{1}{j+1}$$

so that

$$\int_0^{1/2} |D_N(x)|\, dx \geq \frac{2}{\pi^2} \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{N} \right)$$

$$\boxed{\int_{-1/2}^{1/2} |D_N(x)|\, dx \to \infty \quad \text{as } N \to \infty}$$

Using this property one can show that there are continuous functions for which $f_{(N)}(0) \to \infty$.

3.6 Fejer kernel, completeness of trigonometric basis $e^{2\pi i k x}, k \in \mathbb{Z}$

In order to obtain nicer convergence properties we use smaller factors for the higher frequencies: We define the **Fejer kernel** $\sigma_N \in \mathscr{T}_N$ by

$$\sigma_N(x) = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1} \right) \cdot e^{2\pi i k x} \tag{40}$$

Note that the factor $1 - \frac{|k|}{N+1}$ decreases linearly from 1 to 0 as $k = 0, \ldots, N+1$.

Therefore (32), (33) give for the convolution $q_N = f * \sigma_N$

$$q_N(x) = \sum_{k=-N} \left(1 - \frac{|k|}{N+1} \right) \hat{f}_k e^{2\pi i k x}$$

For each k we have $1-\frac{|k|}{N+1}\to 1$ as $N\to\infty$. So we can hope that $q_N(x)\to\sum_{k=-\infty}^{\infty}\hat{f}_k e^{2\pi ikx}=f(x)$ as $N\to\infty$.

The Fejer kernel is a 1-periodic real function. What does the graph look like?

We claim

$$\sigma_N(x):=\sum_{k=-N}^{N}\left(1-\frac{|k|}{N+1}\right)u^{(k)}=\begin{cases}N+1 & \text{for } x\in\mathbb{Z}\\ \dfrac{\sin^2((N+1)\pi x)}{(N+1)\cdot\sin^2(\pi x)} & \text{otherwise}\end{cases}$$

Proof: For nonzero $x\in[-\frac{1}{2},\frac{1}{2}]$ let $r:=e^{2\pi ix}$. We use

$$(1+r+\cdots+r^N)^2=1+2r+\cdots+Nr^{N-1}+(N+1)r^N+Nr^{N+1}+\cdots+1\cdot r^{2N}$$

$$\underbrace{r^{-N}\cdot\overbrace{\left(\frac{r^{N+1}-1}{r-1}\right)^2}}_{\left(\frac{r^{\frac{N}{2}+1}-r^{-\frac{N}{2}}}{r-1}\right)^2}=r^{-N}+2r^{-N+1}+\cdots+Nr^{-1}+(N+1)r^0+Nr^1+\cdots+1\cdot r^N$$

Hence with $r:=e^{2\pi ikx}$

$$\sigma_N(x)=\frac{1}{N+1}\left[r^{-N}+2r^{-N+1}+\cdots+Nr^{-1}+(N+1)r^0+Nr^1+\cdots+1\cdot r^N\right]$$
$$=\frac{1}{N+1}\left(\frac{r^{\frac{N}{2}+1}-r^{-\frac{N}{2}}}{r-1}\right)^2=\frac{1}{N+1}\left(\frac{r^{\frac{N+1}{2}}-r^{-\frac{N+1}{2}}}{r^{\frac{1}{2}}-r^{-\frac{1}{2}}}\right)^2\quad\square$$

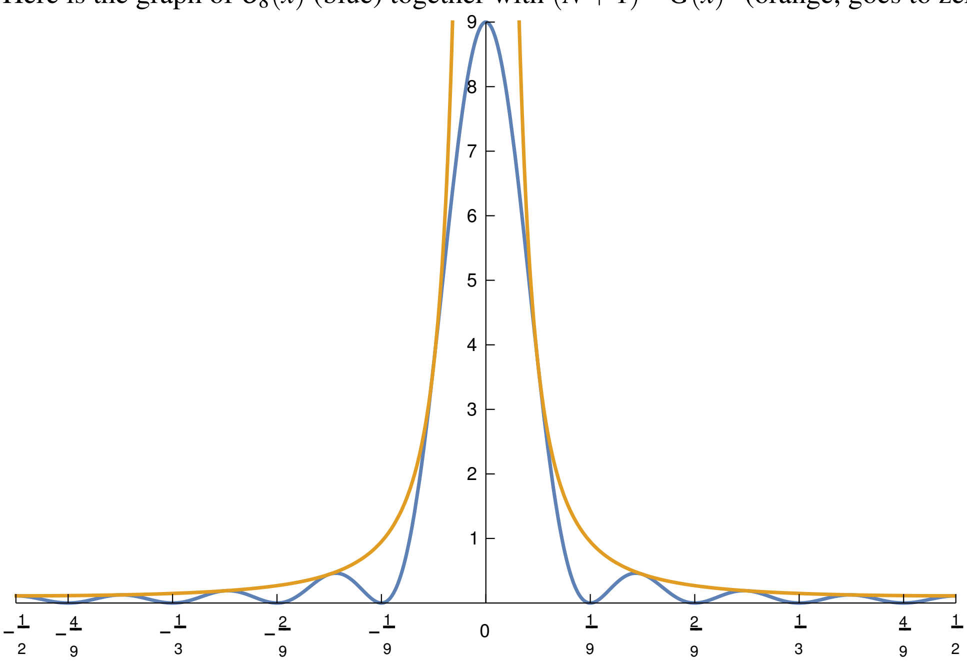
Using the sinc function we can write σ_N

$$\sigma_N(x)=(N+1)\frac{\operatorname{sinc}^2((N+1)\pi x)}{\operatorname{sinc}^2(\pi x)}$$

For $x\neq 0$ we get with $G(x)$ from (39)

$$\sigma_N(x)=\frac{1}{N+1}\cdot\overbrace{\frac{1}{\pi^2x^2\operatorname{sinc}^2(\pi x)}}^{G(x)^2}\cdot\sin^2((N+1)\pi x)$$

Here is the graph of $\sigma_8(x)$ (blue) together with $(N+1)^{-1}G(x)^2$ (orange, goes to zero as $N\to\infty$):



Note that we have

1. $\sigma_N(x) \geq 0$
2. Since in (40) the coefficient for $k=0$ is 1: $\int_{-1/2}^{1/2} \sigma_N(x)dx = 1$.
3. We have $\sigma_N(x) \leq \frac{1}{N+1} \cdot \left(\frac{2}{x}\right)^2$, so for $|x| \geq \delta$ we have $\sigma_N(x) \leq \frac{1}{N+1} \cdot \frac{4}{\delta^2} \to 0$ as $N \to \infty$.

As σ_N satisfies (P1), (P2), (P3) we obtain from Lemma 2 that $q_N = \sigma_N * f$ converges uniformly to f.

Hence we obtain

Theorem 3. *Assume that f is 1-periodic and continuous. For any $\varepsilon > 0$ there exists a trigonometric polynomial $q_n \in \mathscr{T}_n$ such that*

$$\max_x |f(x) - q_n(x)| \leq \varepsilon.$$

Remarks:

- We did NOT prove that $\max \left|f - f_{(N)}\right| \to 0$ as $N \to \infty$. There are continuous functions f where $f_{(N)}$ does not uniformly converge to f.
- However, Theorem 3 shows that the mean square error $\left\|f - f_{(N)}\right\|$ converges to zero: Since $f_{(N)}$ is the best approximation in $\mathscr{T}_N$ we have $\left\|f - f_{(N)}\right\| \leq \|f - q\|$ for any $q \in \mathscr{T}_N$ and hence

$$\left\|f - f_{(N)}\right\|^2 \leq \|f - q_N\|^2 = \int_0^1 |f(x) - q_N(x)|^2 \, dx \leq (\max |f - q_N|)^2 \to 0 \qquad \text{as } N \to \infty$$

Now we want to consider more general functions.

For a **piecewise continuous function** f with finitely many jumps at points $z_1, \ldots, z_k$ we can construct a continuous function f_ε such that $\|f - f_\varepsilon\| \leq \varepsilon$: This can be done by cutting out intervals $[z_j - \delta, z_j + \delta]$ and replacing the function f with a straight line on these intervals.

This means: For any given ε we can first construct a function f_ε with $\|f - f_\varepsilon\| \leq \varepsilon$. Then we can use Theorem 3 to construct for f_ε a function q_n such that $|f_\varepsilon(x) - q_n(x)| \leq \varepsilon$ and hence

$$\|f_\varepsilon - q_n\|^2 = \int_0^1 |f_\varepsilon(x) - q_n(x)|^2 \, dx \leq \int_0^1 \varepsilon^2 dx = \varepsilon^2$$

so that we obtain

$$\|f - q_n\| \leq \|f - f_\varepsilon\| + \|f_\varepsilon - q_n\| \leq \varepsilon + \varepsilon = 2\varepsilon$$

i.e., we can find trigonometric functions q_n such that the L^2-error $\|f - q_n\|$ is arbitrarily small.

The same argument works for a **square integrable function** $f \in L^2$, i.e.,

$$\int_0^1 |f(x)|^2 \, dx < \infty$$

(in the sense of a Lebesgue integral). For any $\varepsilon > 0$ it is possible to construct a continous function f_ε with $\|f - f_\varepsilon\| \leq \varepsilon$ (see e.g. Theorem 3.14 in [Rudin: Real and Complex Analysis]).

This implies that any square integrable function can be approximated by trigonometric polynomials in the mean square sense:

Theorem 4. *Assume that f satisfies $\int_0^1 |f(x)|^2 \, dx < \infty$ (in the Lebesgue sense). For any $\varepsilon > 0$ there exists a trigonometric polynomial $q_n \in \mathscr{T}_n$ such that $\|f - q_n\| \leq \varepsilon$.*

We now obtain the proof of the convergence result (T1) in section 3.3:

Corollary 5. *Assume that f satisfies $\int_0^1 |f(x)|^2 \, dx < \infty$. Let $f_{(N)}(x) = \sum_{k=-N}^{k=N} \hat{f}_k e^{2\pi i k x}$. Then we have*

$$\left\|f - f_{(N)}\right\| \to 0 \qquad \text{as } N \to \infty \tag{41}$$

$$\int_0^1 |f(x)|^2 \, dx = \sum_{k=-\infty}^{\infty} \left|\hat{f}_k\right|^2 \qquad \textbf{\textit{“Parseval identity”}} \tag{42}$$

Proof. By Theorem 4 we can find for any given $\varepsilon > 0$ a function $q_N \in \mathscr{T}_N$ such that $\|f - q_N\| < \varepsilon$. Since $f_{(N)}$ is the best approximation of f in $\mathscr{T}_N$ (see (6)) we have

$$\left\|f - f_{(N)}\right\| \le \|f - q_N\| \le \varepsilon$$

which gives (41). From (19) we have

$$\|f\|^2 = \left\|f - f_{(N)}\right\|^2 + \sum_{k=-N}^{N} \left|\hat{f}_k\right|^2$$

□

As $N \to \infty$ we have $\left\|f - f_{(N)}\right\| \to 0$. Hence $\sum_{k=-N}^{N} \left|\hat{f}_k\right|^2$ converges to $\|f\|^2$ for $N \to \infty$.

For functions $f, g \in L^2$ we have the inner product $(f,g) = \int_0^1 f(x)\overline{g(x)}dx$ and norm $\|f\| = (f,f)^{\frac{1}{2}}$.

Let us denote by $\vec{\hat{f}} = \left(\hat{f}_k\right)_{k\in\mathbb{Z}}$ the "infinite vector" of all Fourier coefficients.

For $\vec{\hat{f}}$, $\vec{\hat{g}}$ we have the inner product $\left(\vec{\hat{f}}, \vec{\hat{g}}\right) = \sum_{k=-\infty}^{\infty} \hat{f}_k \overline{\hat{g}_k}$ and the norm $\left\|\vec{\hat{f}}\right\| = \left(\vec{\hat{f}}, \vec{\hat{f}}\right)^{\frac{1}{2}}$.

The Parseval identity states that $\|f\| = \left\|\vec{\hat{f}}\right\|$. We have seen in (3) that we can express an inner product (u,v) in terms of norms $\|u+v\|$, $\|u+iv\|$, $\|u\|$, $\|v\|$. Therefore the Parseval identity implies

$$\text{for all } f,g \in L^2: \qquad (f,g) = \left(\vec{\hat{f}}, \vec{\hat{g}}\right)$$

$$\int_0^1 f(x)\overline{g(x)}dx = \sum_{k=-\infty}^{\infty} \hat{f}_k \overline{\hat{g}_k}$$

3.7 Summary: Fourier series for 1-periodic functions

Here we summarize the **key properties of the Fourier series** for **1-periodic functions** f, g:

$$\boxed{\begin{array}{lcll} \hat{f}_k = \int_0^1 f(x) e^{-2\pi ikx} dx & \Leftrightarrow & f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{2\pi ikx} & \textbf{Inversion} \\ \int_0^1 |f(x)|^2 dx & = & \sum_{k=-\infty}^{\infty} \left|\hat{f}_k\right|^2 & \textbf{Parseval} \\ q(x) = \int_0^1 f(t) g(x-t) dt & \Leftrightarrow & \hat{q}_k = \hat{f}_k \, \hat{g}_k & \textbf{Convolution} \end{array}} \tag{43}$$

- The statement $f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{2\pi ikx}$ is interpreted as a limit as stated in (T1), (T2), (T3) in section 3.3.
- The Parseval identity actually holds for inner products: $\int_0^1 f(x)\overline{g(x)}dx = \sum_{k=-\infty}^{\infty} \hat{f}_k \overline{\hat{g}_k}$.

3.8 Fourier series for L-periodic functions

So far we considered 1-periodic functions. But all results carry over if we have a different period:

Consider a function $f(x)$ with period $L > 0$, i.e., we have $f(x+L) = f(x)$ for all $x \in \mathbb{R}$. Then the function $F(x) := f(Lx)$ is 1-periodic. Using our previous results for F we have

$$f(\underbrace{Lx}_{t}) = F(x) = \sum_{k=-\infty}^{\infty} \hat{F}_k e^{2\pi ikx} = \sum_{k=-\infty}^{\infty} \hat{F}_k e^{2\pi ikt/L}$$

With the change of variables $t = Lx$ we obtain

$$\hat{f}_k = \hat{F}_k = \int_{x=0}^{1} f(\underbrace{Lx}_{t}) e^{-2\pi ikx} \underbrace{dx}_{L^{-1}dt} = \int_{t=0}^{L} f(t) e^{-2\pi ikt/L} L^{-1} dt$$

$$
\boxed{\begin{array}{lcll}
\hat{f}_k = L^{-1}\int_0^L f(x)e^{-2\pi ikx/L}dx & \Leftrightarrow & f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{2\pi ikx/L} & \textbf{Inversion} \\
L^{-1}\int_0^L |f(x)|^2\, dx & = & \sum_{k=-\infty}^{\infty} \left|\hat{f}_k\right|^2 & \textbf{Parseval} \\
q(x) = L^{-1}\int_0^L f(t)g(x-t)dt & \Leftrightarrow & \hat{q}_k = \hat{f}_k\, \hat{g}_k & \textbf{Convolution}
\end{array}} \tag{44}
$$

Note:

1. Because of the change of variables we obtain $L^{-1}\int_0^L \cdots$ in place of $\int_0^1 \cdots$ on the left hand side.
2. f is written as a linear combination of the functions $e^{2\pi i\xi_k x}$ with frequencies $\xi_k = \dfrac{k}{L}$, $k \in \mathbb{Z}$.

3.9 Pointwise convergence

We now prove (T2) in section 3.3.

Assume $f \in PW^1$. Since $f_{(N)} = f * D_N$ and D_N is even we have

$$
f_{(N)}(x_0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} D_N(t)f(x_0-t)dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} D_N(t)f(x_0+t)dt = \underbrace{\int_{-\frac{1}{2}}^{0} D_N(t)f(x_0+t)dt}_{I_N^-} + \underbrace{\int_0^{\frac{1}{2}} D_N(t)f(x_0+t)dt}_{I_N^+}
$$

We claim that

$$
I_N^- \to \tfrac{1}{2}f(x_0-0), \quad I_N^+ \to \tfrac{1}{2}f(x_0+0) \quad \text{as } N \to \infty
$$

We define for $t \in (0, \frac{1}{2}]$

$$
g(t) := \frac{f(x_0+t) - f(x_0+0)}{\sin(\pi t)}
$$

This function is piecewise continuous on $(0, \frac{1}{2}]$, and the limit at 0 is

$$
\lim_{\substack{t\to 0\\ t>0}} g(t) = \lim_{\substack{t\to 0\\ t>0}} \frac{f(x_0+t) - f(x_0+0)}{t} \cdot \frac{t}{\sin(\pi t)} = D_+f(x_0) \cdot \frac{1}{\pi}
$$

with the 1-sided derivative from the right $D_+f(x_0)$. Let us define $g(-t) := -g(t)$ so that g is an odd function on $[-\frac{1}{2}, \frac{1}{2}]$.

Since $\int_0^{\frac{1}{2}} D_N(t)dt = \frac{1}{2}$ we obtain for the second integral

$$
\begin{aligned}
I_N^+ - \tfrac{1}{2}f(x_0+0) &= \int_0^{\frac{1}{2}} D_N(t)f(x_0+t)dt - \tfrac{1}{2}f(x_0+0) = \int_0^{\frac{1}{2}} [f(x_0+t) - f(x_0-0)]\, D_N(t)dt \\
&= \int_0^{\frac{1}{2}} \frac{f(x_0+t) - f(x_0-0)}{\sin(\pi t)} \sin\left((2N+1)\pi t\right) dt = \int_0^{\frac{1}{2}} g(t)\sin\left((2N+1)\pi t\right) dt \\
&= \tfrac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} g(t)\sin\left((2N+1)\pi t\right) dt = \tfrac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} g(t)\frac{1}{2i}\left(e^{2\pi iNt}e^{\pi it} - e^{-2\pi iNt}e^{-\pi it}\right) dt \\
&= \frac{1}{4i}\left[\hat{F}_{-N} - \hat{G}_N\right]
\end{aligned}
$$

where $F(t) := g(t)e^{\pi it}$ and $G(t) := g(t)e^{-\pi it}$. These functions are piecewise continuous and hence in L^2, so (22) gives $\hat{F}_{-N} \to 0$ and $\hat{G}_N \to 0$ as $N \to \infty$.

In the same way we can prove $I_N^- - \frac{1}{2}f(x_0-0) \to 0$.

3.10 Uniform convergence

We now prove (T3) in section 3.3.

Assume $f \in PW^1$ without jumps. Then we have a derivative $g := f' \in PW^0$ with Fourier coefficients $\hat{g}_k$. By the Bessel inequality (20) $\sum_{k=-\infty}^{\infty} |\hat{g}_k|^2 \le \|g\|^2$.

Using integration by parts we obtain using that $f(t)$ and $e^{-2\pi ikt}$ are continuous and 1-periodic

$$\hat{g}_k = \int_0^1 f'(t)e^{-2\pi ikt}dt = \underbrace{\left[f(t)e^{-2\pi ikt}\right]_0^1}_{0} - \int_0^1 f(t)(-2\pi ik)e^{-2\pi ikt}dt \tag{45}$$

$$= 2\pi ik\hat{f}_k$$

We obtain for the maximal error

$$\max_{x\in[0,1]} \left|f(x) - f_{(N)}(x)\right| = \max_{x\in[0,1]} \underbrace{\left|\sum_{|k|>N} \hat{f}_k e^{2\pi ikx}\right|}_{\le \sum_{|k|>N} \underbrace{\left|\hat{f}_k e^{2\pi ikx}\right|}_{\left|\hat{f}_k\right|}} \le \sum_{|k|>N} \left|\hat{f}_k\right|$$

For real sequences u_j, v_j we have the Cauchy Schwarz inequality:

$$\left|\sum_{k=1}^{\infty} u_k v_k\right| \le \left(\sum_{k=1}^{\infty} u_k^2\right)^{1/2} \left(\sum_{k=1}^{\infty} v_k^2\right)^{1/2}$$

yielding

$$\sum_{|k|>N} \left|\hat{f}_k\right| = \sum_{|k|>N} \left|\hat{f}_k\right| |k| \cdot |k|^{-1} \le \underbrace{\left(\sum_{|k|>N} \left|\hat{f}_k\right|^2 |k|^2\right)^{\frac{1}{2}}}_{A} \underbrace{\left(\sum_{|k|>N} |k|^{-2}\right)^{\frac{1}{2}}}_{B}$$

Note that $\left|\hat{f}_k\right| \cdot |k| = \frac{1}{2\pi} |\hat{g}_k|$, hence $\sum_{|k|>N} \left|\hat{f}_k\right|^2 |k|^2 \le \frac{1}{4\pi^2} \sum_{k=-\infty}^{\infty} |\hat{g}_k|^2$ and $A \le \frac{1}{2\pi} \|g\|$.

For B we use

$$\sum_{k=N+1}^{\infty} |k|^{-2} \le \int_{x=N}^{\infty} x^{-2}dx = N^{-1}, \qquad B \le (2N^{-1})^{\frac{1}{2}} = \sqrt{2}N^{-\frac{1}{2}}$$

yielding

$$\max_{x\in[0,1]} \left|f(x) - f_{(N)}(x)\right| \le \frac{1}{\pi\sqrt{2}} \|g\| \cdot N^{-1/2} \to 0 \qquad \text{as } N \to \infty \tag{46}$$

Remark: Here we only used that $g = f' \in L^2$.
For $f \in PW^1$ we can obtain a better result if we additionally assume that **each f_j' is piecewise monotonic** (or one can assume $f \in PW^2$, i.e., for each piece f_j' , f_j'' are continuous). Then one can show $\left|\hat{f}_k\right| \le c\,|k|^{-2}$ yielding

$$\max_{x\in[0,1]} \left|f(x) - f_{(N)}(x)\right| \le C \cdot N^{-1} \tag{47}$$

This is the actual convergence rate which one observes for piecewise smooth functions without jumps.

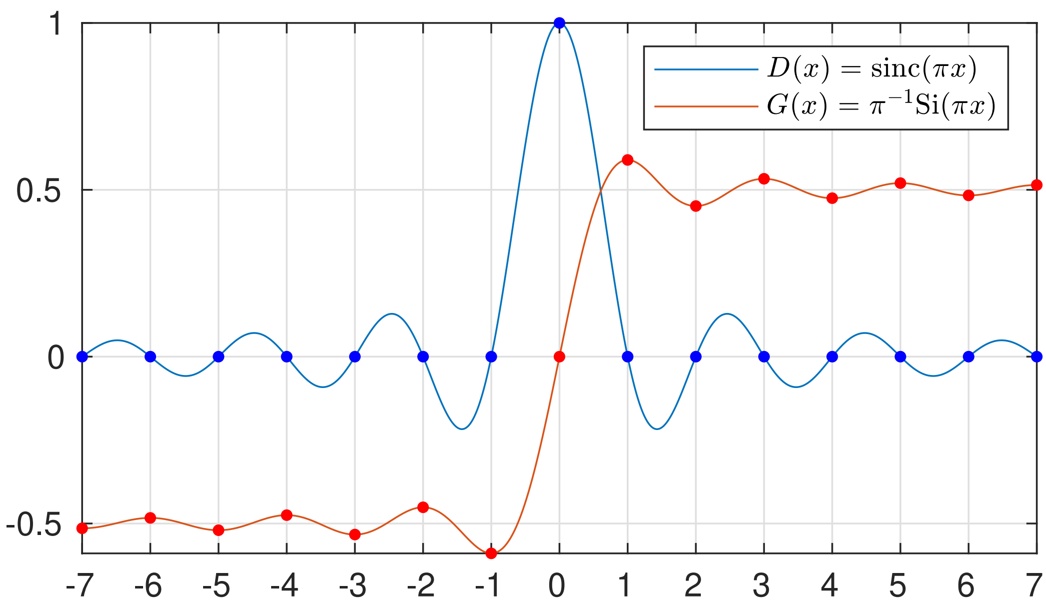
3.11 $D(x) = \operatorname{sinc}(\pi x)$ and its antiderivative $G(x) = \frac{1}{\pi}\operatorname{Si}(x)$

For the function $\operatorname{sinc} x := \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$ we consider the antiderivative

$$\operatorname{Si}(x) := \int_0^x \operatorname{sinc} t \, dt$$

This is called the "**sine integral function**" $\operatorname{Si}(x)$. This cannot be expressed in terms of elementary functions. In Matlab you can use **`sinint(x)`**.

We will consider the function $\boxed{D(x) := \operatorname{sinc}(\pi x)}$ and its antiderivative $\boxed{G(x) := \frac{1}{\pi}\operatorname{Si}(\pi x)}$. Here are their graphs:



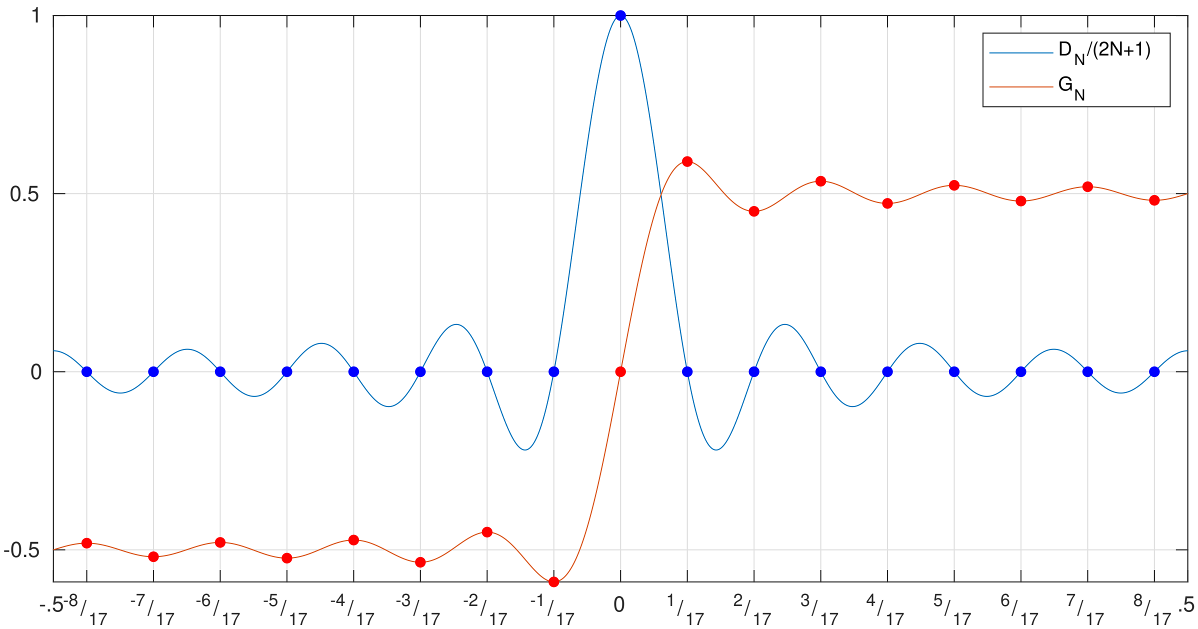
We have the following **properties of** $G(x)$**:**

- one can show that $\int_{-\infty}^{\infty} D(x)dx = 1$. As $D(x) = D(-x)$ we have $\int_0^{\infty} D(x)dx = \frac{1}{2}$.
- $D(x)$ is positive on $(0,1)$, negative on $(1,2)$, positive on $(2,3)$ …. . $D(x)$ has zeros at $1,2,3,\ldots$
  Hence $G(x)$ has local maximum at 1, a local minimum at 2, a local maximum at 3, …
- let $\boxed{a_k := \int_k^{k+1} \operatorname{sinc}(\pi x)\, dx}$. Note that $a_0 > 0$, $a_1 < 0$, $a_2 > 0$, etc.
  As $\int_0^1 \sin(\pi x)dx = \frac{2}{\pi}$ and $|\operatorname{sinc}(\pi x)| \leq \frac{1}{\pi x}$ we have $|a_k| \leq \frac{2}{\pi^2} \cdot \frac{1}{k}$. We also have $|a_{k+1}| \leq |a_k|$.
- We have $G(k) = a_0 + a_1 + \cdots + a_{k-1}$.
  As $k \to \infty$ we obtain the alternating series $a_0 + a_1 + a_2 + \cdots = \frac{1}{2}$.
  Since we have an alternating series with $|a_{k+1}| \leq |a_k|$ and $a_k \to 0$ as $k \to \infty$ the "alternating series theorem" gives:
  $G(k) = a_0 + \cdots + a_{k-1}$ converges to $\frac{1}{2}$ as $k \to \infty$, and the error satisfies $\left|G(k) - \frac{1}{2}\right| \leq |a_k| \leq \frac{2}{\pi^2} \cdot \frac{1}{k}$

  We have for actually for all $x > 0$ that $\boxed{\left|G(x) - \frac{1}{2}\right| \leq \frac{2}{\pi^2} \cdot \frac{1}{x}}$.

3.12 The Dirichlet kernel $D_N(x)$ and its antiderivative $G_N(x)$

We consider the Dirichlet kernel $D_N(x)$ (1-periodic) and its antiderivative $G_N(x) := \int_0^x D_N(t)dt$. Here are the graphs of $D_N/(2N+1)$ and G_N for $N = 8$:



We have the following **properties of $G_N(x)$:**

- we saw that $\int_{-\frac{1}{2}}^{\frac{1}{2}} D_N(x)dx = 1$. As $D_N(-x) = D_N(x)$ we have $\int_0^{\frac{1}{2}} D(x)dx = \frac{1}{2}$.
- $D_N(x)$ is positive on $(0, \frac{1}{2N+1})$, negative on $(\frac{1}{2N+1}, \frac{2}{2N+1})$, positive on $(\frac{2}{2N+1}, \frac{3}{2N+1})$ …. . $D_N(x)$ has zeros at $\frac{1}{2N+1}, \frac{2}{2N+1}, \frac{3}{2N+1}, \ldots$
  Hence $G_N(x)$ has local maximum at $\frac{1}{2N+1}$, a local minimum at $\frac{2}{2N+1}$, a local maximum at $\frac{3}{2N+1}$, …
- let $\boxed{a_k := \int_{k/(2N+1)}^{(k+1)/(2N+1)} D_N(x)\,dx}$ for $k = 0, \ldots, N-1$, $a_N := \int_{N/(2N+1)}^{1/2} D_N(x)dx$. Note that $a_0 > 0$, $a_1 < 0$, $a_2 > 0$, etc.
  As $\int_0^1 \sin(\pi x)dx = \frac{2}{\pi}$ and $|D_N(x)| \le \frac{1}{2x}$ we have $|a_k| \le \frac{1}{\pi} \cdot \frac{1}{k}$ for $k = 1, 2, \ldots$. We also have $|a_{k+1}| \le |a_k|$.
- We have $G_N(\frac{k}{2N+1}) = a_0 + a_1 + \cdots + a_{k-1}$ for $k = 1, \ldots, N$ and $a_0 + a_1 + \cdots + a_N = \frac{1}{2}$.
  Since we have an alternating series with $|a_{k+1}| \le |a_k|$ the "alternating series theorem" gives:
  For $k = 0, \ldots, N$ the error satisfies $\left|G_N(\frac{k}{2N+1}) - \frac{1}{2}\right| \le |a_k| \le \frac{1}{\pi} \cdot \frac{1}{k}$, i.e. for $x = \frac{k}{2N+1}$ we have $\left|G_N(x) - \frac{1}{2}\right| \le \frac{1}{\pi} \cdot \frac{1}{(2N+1)x}$.
  We have actually for all $x \in (0, \frac{1}{2}]$ that $\boxed{\left|G_N(x) - \frac{1}{2}\right| \le \frac{1}{\pi} \cdot \frac{1}{(2N+1)x}}$.
- $\boxed{G_N\left(\frac{s}{2N+1}\right) = G(s) + O(N^{-2})}$: We get with the change of variables $z = (2N+1)t$

$$\begin{aligned} G_N(\frac{s}{2N+1}) &= (2N+1)\int_{t=0}^{s/(N+1)} \frac{\operatorname{sinc}((2N+1)\pi t)}{\operatorname{sinc}(\pi t)}dt = \int_{z=0}^{s} \frac{\operatorname{sinc}(\pi z)}{\operatorname{sinc}(\pi z/(2N+1))}dz \\ &= \pi^{-1}\operatorname{Si}(\pi s) + O(N^{-2}) \end{aligned}$$

  since $\operatorname{sinc}(h) = 1 + O(h^2)$ as $h \to 0$.

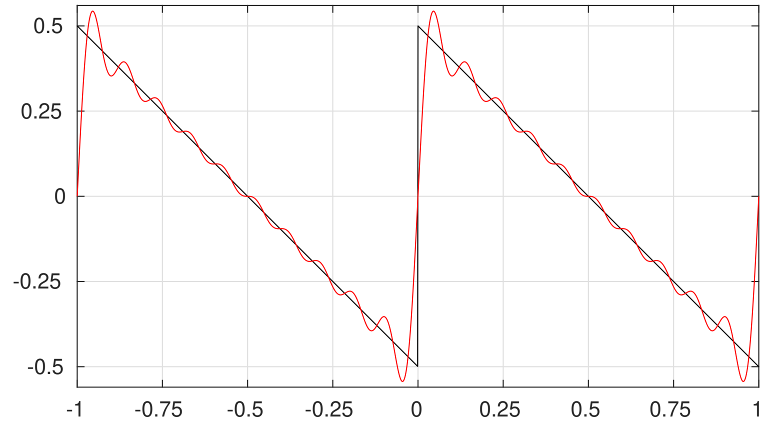
3.13 Gibbs phenomenon: behavior of error $f_{(N)}(x) - f(x)$ for functions with jumps and kinks

Consider a function $f \in PW^1$ with jumps. We want to investigate the behavior of the error

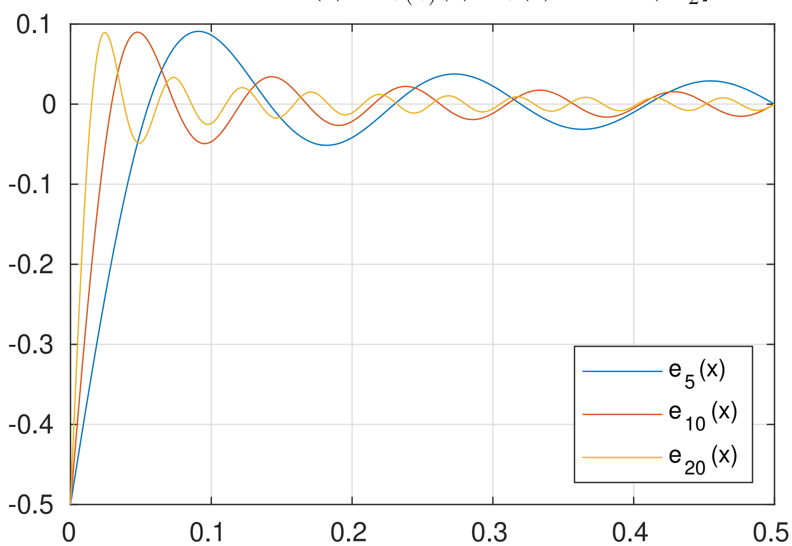
$$e_N(x) := f_{(N)}(x) - f(x)$$

Gibbs phenomenon for $g(x) = \frac{1}{2} - x$ on $[0,1]$

We first consider the 1-periodic function g with $\boxed{g(x) = \frac{1}{2} - x}$ for $x \in [0,1)$. This function has a jump of size $J = 1$ at 0. Here is g and the Fourier approximation $g_{(10)}$:

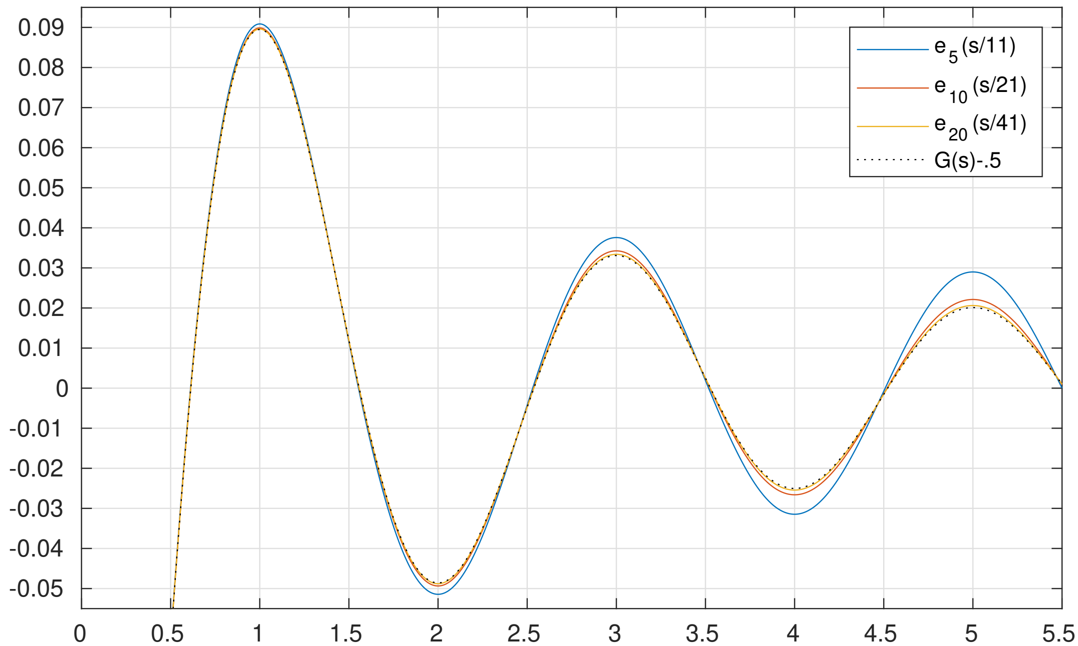


Here are the error functions $e_N(x) := g_{(N)}(x) - g(x)$ for $x \in (0, \frac{1}{2}]$ and $N = 5, 10, 20$:



We observe:

- for each N we have oscillations which decay like $\frac{c}{x}$ away from the jump at 0
- as N increases the amplitude of the oscillations at a point x decreases like C_x/N
- the oscillations have period $\frac{1}{N+\frac{1}{2}}$
- the functions $e_N\left(\frac{s}{2N+1}\right)$ are almost the same for $N = 5, 10, 20$, and almost the same as $G(s) - \frac{1}{2}$:



We will prove: Let d_x denotes the closest distance from x to the jump, then

$$|e_N(x)| \leq \frac{1}{\pi} \cdot \frac{J}{(2N+1)d_x} \tag{48}$$

Close to the jump at z_0=0 we have

$$\boxed{e_N\left(z_0 + \frac{s}{2N+1}\right) \approx J \cdot \left[G(s) - \tfrac{1}{2}\operatorname{sign}(s)\right]} \tag{49}$$

where $G(s) := \pi^{-1}\operatorname{Si}(\pi s)$. For this specific function g this holds with an error term of $O(N^{-2})$. But for general functions $f \in PW^1$ which have several jumps, or different slopes to the left and right of the jump at z_0 (49) holds with an error term of $O(N^{-1})$.

This means to the right of the jump we have

$$e_N(z_0+0) = -\tfrac{1}{2}J, \qquad e_N(z_0 + \frac{1}{2N+1}) \approx .0895J, \qquad e_N(z_0 + \frac{2}{2N+1}) \approx -.0486J$$

To the left of the jump we have the opposite signs.

Claim: $e_N(x) = G_N(x) - \frac{1}{2}$ for $x \in (0, \frac{1}{2})$.

Proof: Note that for $s \in [-1,1]$ we have $g(s) = -s - \frac{1}{2} + \begin{cases} 1 & s > 0 \\ 0 & s < 0 \end{cases}$.

Hence for $x,t \in (0,1)$ we have $g(x-t) = t - x - \frac{1}{2} + \begin{cases} 1 & t < x \\ 0 & t > x \end{cases}$ and

$$\begin{aligned} g_{(N)}(x) &= \int_{t=0}^{1} D_N(t) g(x-t)dt = \int_{t=0}^{1} D_N(t)(t-x-\tfrac{1}{2})dt + \int_{t=0}^{x} D_N(t)dt \\ &= \underbrace{\int_{t=0}^{1} D_N(t)(t-\tfrac{1}{2})dt}_{0} - x \underbrace{\int_{t=0}^{1} D_N(t)dt}_{1} + G_N(x) \end{aligned}$$

$$g_{(N)}(x) - \underbrace{(\tfrac{1}{2} - x)}_{g(x)} = G_N(x) - \tfrac{1}{2}$$

Gibbs phenomenon for $f \in PW^1$

For a piecewise continuous differentiable function $f \in PW^1$ with jumps $J_k := f(z_k+0) - f(z_k-0)$ at z_k, $k = 1,\ldots,K$ we see that

$$F(x) := f(x) - J_1 \cdot g(x-z_1) - \cdots - J_K \cdot g(x-z_K)$$

does not have jumps and $F \in PW^1$. If each f_j' is piecewise monotonic this also holds for F and (47) gives

$$\max_x \left|F_{(N)}(x) - F(x)\right| \le cN^{-1}$$

and for each term $J_\ell g(x-z_\ell)$ we get (48), (49). For $f(x) = F(x) + J_1 \cdot g(x-z_1) + \cdots + J_K \cdot g(x-z_K)$ we get the sum of these terms.

Gibbs phenomenon for function with "kinks"

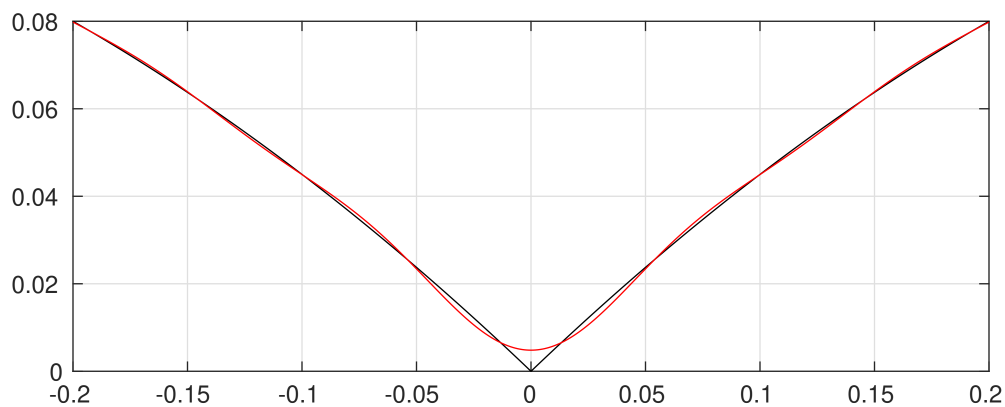
Assume that f is a piecewise smooth function without jumps. In this case we have (47)

$$\max_x \left|f_{(N)}(x) - f(x)\right| \le CN^{-1}$$

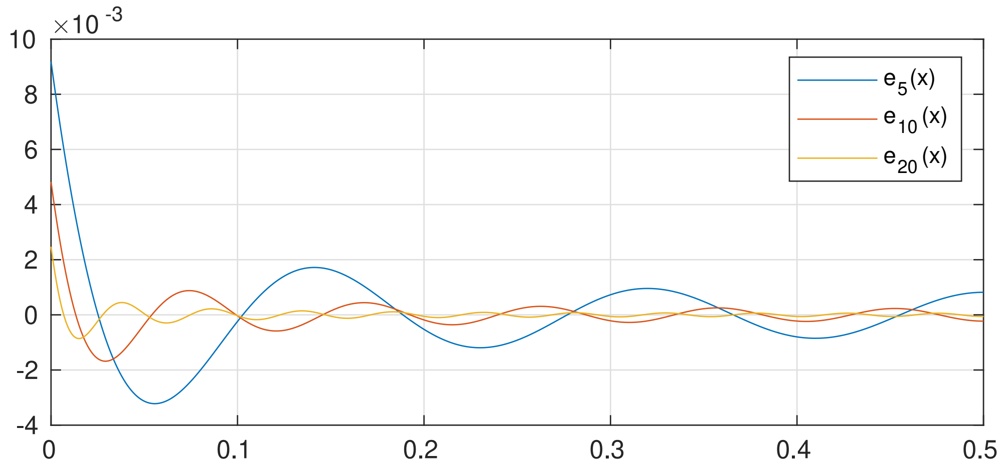
The "bad points" are the "kinks" z_j where f' has a jump of J: denoting the right and left sided derivative by D_+, D_-

$$J := D_+f(z_j) - D_-f(z_j)$$

Example: Let f denote the 1-periodic function with $f(x) = (x-x^2)/2$. Here $J = D_+f(0) - D_-f(0) = \frac{1}{2} - (-\frac{1}{2}) = 1$. Here are f and $f_{(10)}$ around the kink at 0:

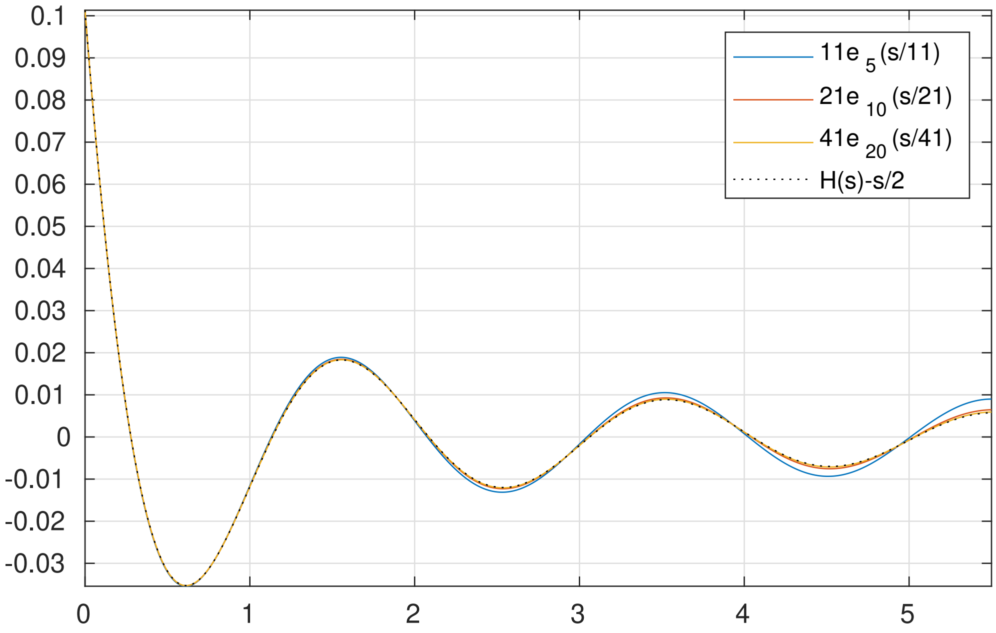


Here are the errors $e_N := f_{(N)} - f$ for $N = 5, 10, 20$:



At a kink z_j we have an error $\left|f_{(N)}(z_j) - f(z_j)\right| \le cN^{-1}$ whereas at a point x away from kinks we have $\left|f_{(N)} - f\right| \le C_x N^{-2}$. Here C_x decays like $1/d_x$ where d_x is the distance to the closest kink.

We notice that $(2N+1)e_N(z_0+\frac{s}{2N+1})$ look similar for $N=5,10,20$



Here $H(s)$ is an antiderivative of $G(s)$

$$H(s) := \pi^{-2}\cos(\pi s) + \pi^{-1} s \cdot \mathrm{Si}(\pi s)$$

and we obtain close to a kink z_0 for the error $e_N := f_{(N)} - f$

$$e_N\left(z_0 + \frac{s}{2N+1}\right) \approx \frac{J}{2N+1}\cdot\left[H(s) - \tfrac{1}{2}|s|\right]$$

In particular we have $e_N(z_0) \approx \frac{J}{2N+1}\cdot\frac{1}{\pi^2}$.

3.14 Generalized functions, derivatives and the Dirac delta

Functions and generalized functions on $\mathbb{R}$

Let f denote a function defined on the real line $\mathbb{R}$. Let e.g. $f(x)$ denote the temperature in a room along a straight line. For a given position x_0 we have a corresponding temperature value $f(x_0)$ at this point. However, when we try to measure the temperature with a thermometer we will not get the value at a single point. As the thermometer has finite size, we will get a weighted average of temperatures: We have a function $\varphi(x)$ which is nonzero only on some small interval (a,b), and we obtain a weighted average

$$A = \int_{-\infty}^{\infty} f(x)\varphi(x)dx$$

We use the notation $A = \langle f,\varphi\rangle$ which means: "test f with the test function $\varphi(x)$". We assume that **test functions** φ satisfy

- φ and its derivatives $\varphi', \varphi'', \varphi''', \ldots$ exist and are continuous
- φ is nonzero only on some bounded interval

Summary: For a **"classical function"** $f(x)$ we can perform the following two operations:

(A) Sampling at a point $x\in\mathbb{R}$: For any value $x\in\mathbb{R}$ we obtain an output value $y = f(x)\in\mathbb{R}$

(B) Testing with a test function $\varphi(x)$: For any test function $\varphi(x)$ we obtain an output value $Y = \langle f,\varphi\rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)dx$.

We have seen that there are things like the "Dirac Delta" which are not classical functions. Therefore we now define "**generalized functions**" (a.k.a. "distributions"). For a generalized function F the operation (A) may not make sense. However, the operation (B) is always well defined: For any test function φ we obtain an output value $Y = \langle f, \varphi \rangle$.

Example: For a sequence of functions $g^{(n)}$ with (P1), (P2), (P3) we have for any continuous test function φ

$$\left\langle g^{(n)}, \varphi \right\rangle = \int_{-\infty}^{\infty} g^{(n)}(x)\varphi(x)dx \to \varphi(0) \qquad \text{as } n \to \infty$$

So the "limit object" is the generalized function $\boxed{\delta}$ which maps any given test function φ to the output value

$$\langle \delta, \varphi \rangle := \varphi(0)$$

This generalized function is called **Dirac delta**. There is no classical function with this property: we would need a function with integral 1 which is only nonzero in the point $x = 0$.

Similarly we **define** $\boxed{\delta_b}$ **as the Dirac delta concentrated in** $x = b$, i.e.,

$$\boxed{\langle \delta_b, \varphi \rangle = \varphi(b)}$$

We can always define the derivative $G = F'$ **of a generalized function, yielding another generalized function** G**.** For a classical function f we have for any test function using integration by parts

$$\left\langle f', \varphi \right\rangle = \int_a^b f'(x)\varphi(x)dx = \underbrace{[f(x)\varphi(x)]_a^b}_{0} - \int_a^b f(x)\varphi'(x)dx = -\left\langle f, \varphi' \right\rangle$$

where φ is zero outside of the interval (a,b). Therefore **we define the generalized function** F' by

$$\boxed{\left\langle F', \varphi \right\rangle := -\left\langle F, \varphi' \right\rangle}$$

Example 1: Let $F = \delta_3$. Then F' is the generalized function which gives for a test function φ the output value

$$\left\langle F', \varphi \right\rangle = -\left\langle F, \varphi' \right\rangle = -\varphi'(3)$$

Example 2: Consider the **Heaviside function** $H(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$. It does not have a derivative which is a classical function. However, if we consider H as a generalized function we obtain a derivative H': For a test function φ we get

$$\left\langle H', \varphi \right\rangle = -\left\langle H, \varphi' \right\rangle = -\int_{-\infty}^{\infty} H(x)\varphi'(x)dx = -\int_0^{\infty} \varphi'(x)dx = -[\varphi(x)]_0^b = \varphi(0)$$

Hence we have that the derivative of the Heaviside function is the Dirac Delta:

$$\boxed{H' = \delta_0}$$

Example 3: Consider a **piecewise differentiable function**

$$f(x) = \begin{cases} f_1(x) & \text{for } x < A \\ f_2(x) & \text{for } x \geq A \end{cases}$$

where $f_1(x)$ has a continuous derivative on $(-\infty, A]$ and $f_2(x)$ has a continuous derivative on $[A, \infty)$. Let $g(x)$ denote the "piecewise derivative"

$$g(x) := \begin{cases} f_1'(x) & \text{for } x < A \\ f_2'(x) & \text{for } x \geq A \end{cases}$$

What is the derivative of f' as a generalized function? For a test function φ we have an interval (a,b) containing A and all points x with $\varphi(x) \neq 0$. Then

$$\begin{aligned}\langle f', \varphi \rangle = -\langle f, \varphi' \rangle &= -\int_a^A f_1(x)\varphi'(x)dx - \int_A^b f_2(x)\varphi'(x)dx \\ &= \underbrace{[-f_1(x)\varphi(x)]_a^A}_{-f_1(A)\varphi(A)} + \int_a^A f_1'(x)\varphi(x)dx + \underbrace{[-f_2(x)\varphi(x)]_A^b}_{f_2(A)\varphi(A)} + \int_A^b f_2'(x)\varphi(x)dx \\ &= \int_{-\infty}^{\infty} g(x)\varphi(x)dx + (f_2(A) - f_1(A))\,\varphi(A)\end{aligned}$$

yielding

$$\boxed{f' = g + J \cdot \delta_A \qquad \text{with the "jump size" } J := f(A+0) - f(A-0)} \tag{50}$$

Periodic generalized functions

We can now define **1-periodic generalized functions**. E.g., the **1-periodic Dirac delta** $\boxed{\delta_{\text{per}}}$ is concentrated at the points $j \in \mathbb{Z}$. Because of its graph it is also known as the "**Dirac comb**".

Similarly, for $a \in \mathbb{R}$ the 1-periodic generalized function $\boxed{\delta_{\text{per},a}}$ is concentrated at the points $j + a$ with $j \in \mathbb{Z}$. E.g., the generalized function $\delta_{\text{per},\frac{1}{3}}$ is concentrated at the points $\ldots, -\frac{5}{3}, -\frac{2}{3}, \frac{1}{3}, \frac{4}{4}, \frac{7}{3}, \ldots$.

For a 1-periodic classical function f and a 1-periodic test function φ we can define

$$\langle f, \varphi \rangle_{\text{per}} = \int_{\text{period}} f(x)\varphi(x)dx$$

where the integral goes over a full period of length 1 (e.g. from 0 to 1, or from $-\frac{1}{2}$ to $\frac{1}{2}$). For a 1-periodic generalized function F and a 1-periodic test function φ the operation $\langle F, \varphi \rangle_{\text{per}}$ is well defined. E.g., for $F = \delta_{\text{per},\frac{1}{4}}$ and the test function $\varphi(x) = \sin(2\pi x)$ we obtain $\left\langle \delta_{\text{per},\frac{1}{4}}, \varphi \right\rangle_{\text{per}} = \varphi(\frac{1}{4}) = \sin(2\pi\frac{1}{4}) = \sin(\frac{\pi}{2}) = 1$.

3.15 Properties of Fourier coefficients

If we apply certain operations on the function f, how is this reflected in the Fourier coefficients $\hat{f}_k$?

Changing x to $-x$: Let $\boxed{g(x) := f(-x)}$. With the change of variables $s = -t$ we obtain

$$\hat{g}_k = \int_{t=-\frac{1}{2}}^{\frac{1}{2}} f(-t)e^{-2\pi ikt}dt = \int_{s=\frac{1}{2}}^{-\frac{1}{2}} f(s)e^{2\pi kis}(-ds) = \int_{s=-\frac{1}{2}}^{\frac{1}{2}} f(s)e^{2\pi kis}ds = \hat{f}_{-k}$$

hence $\boxed{\hat{g}_k = \hat{f}_{-k}}$.

Shifting the function by a: Let $\boxed{g(x) := f(x-a)}$, i.e., we shift the graph of the function a distance a to the right. With the change of variables $s = t - a$ we obtain

$$\hat{g}_k = \int_0^1 f(t-a)e^{-2\pi ikt}dt = \int_{-a}^{1-a} f(s)e^{-2\pi ik(s+a)}ds = e^{-2\pi ika}\int_0^1 f(s)e^{-2\pi iks}ds = e^{-2\pi ika}\hat{f}_k$$

Hence $\boxed{\hat{g}_k = e^{-2\pi ika}\hat{f}_k}$.

Taking the derivative: Let $\boxed{g(x) := f'(x)}$. Then integration by parts gives

$$\hat{g}_k = \int_0^1 f'(x)e^{-2\pi ikx}dx = \underbrace{\left[f(x)e^{-2\pi ikx}\right]_0^1}_{0} - \int_0^1 f(x)(-2\pi ik)e^{-2\pi ikx}dx = (2\pi ik)\hat{f}_k$$

since $f(x)$ and $e^{-2\pi ikx}$ are 1-periodic functions. Hence $\boxed{\hat{g}_k = 2\pi ik \cdot \hat{f}_k}$.

Recall that functions with $\int_0^1 |f(x)|^2 dx < \infty$ correspond to Fourier coefficients $\hat{f}_k$ with $\sum_{k=-\infty}^{\infty} \left|\hat{f}_k\right|^2 < \infty$.

For a generalized function F we can still define the Fourier coefficients $\hat{F}_k$ by

$$\hat{F}_k = \left\langle F, e^{-2\pi ikx} \right\rangle_{\text{per}}$$

since $\varphi(x) = e^{-2\pi ikx}$ is a 1-periodic test function.

For a generalized function F we can define the "reflected" generalized function $F(-x)$, the shifted generalized function $F(x-a)$, and the derivative F', and **the above rules for the Fourier coefficients still hold**. E.g., for $G = F'$ we have

$$\hat{G}_k = \left\langle F', e^{-2\pi ikx} \right\rangle_{\text{per}} = -\left\langle F, \frac{d}{dx}\left(e^{-2\pi ikx}\right) \right\rangle_{\text{per}} = -\left\langle F, (-2k\pi i)e^{-2\pi ikx} \right\rangle_{\text{per}} = 2k\pi i \left\langle F, e^{-2\pi ikx} \right\rangle_{\text{per}} = 2k\pi i \cdot \hat{F}_k$$

Example: We consider the 1-periodic function f with $f(x) = 1$ on $[0, \frac{1}{2})$ and $f(x) = -1$ on $[-\frac{1}{2}, 0)$. Let us first find the derivative $G = f'$ as a generalized function. The function $f(x)$ is constant, except for jumps of size 2 and -2 at $x = 0$ and $x = \frac{1}{2}$, respectively. Hence we get with (50)

$$G = f' = 0 + 2\delta_{\text{per}} - 2\delta_{\text{per},\frac{1}{2}}$$

Recall that the Fourier coefficients of $g = \delta_{\text{per}}$ are $\hat{g}_k = 1$ for all $k \in \mathbb{Z}$. The generalized function $h = \delta_{\text{per},\frac{1}{2}}$ is obtained by shifting $g = \delta_{\text{per}}$ by $a = \frac{1}{2}$, hence the rule for the shifted function gives

$$\hat{h}_k = e^{-2k\pi i\frac{1}{2}} \cdot \hat{g}_k = (-1)^k \cdot 1$$

Hence $G = 2\delta_{\text{per}} - 2\delta_{\text{per},\frac{1}{2}}$ has the Fourier coefficients

$$\hat{G}_k = 2 - 2 \cdot (-1)^k = \begin{cases} 0 & \text{for even } k \\ 4 & \text{for odd } k \end{cases}$$

Now we can use $G = f'$ to find the Fourier coefficients $\hat{f}_k$: From the derivative rule we have $\hat{G}_k = 2k\pi i \cdot \hat{f}_k$, hence

$$\text{for } k \neq 0: \qquad \hat{f}_k = \frac{\hat{G}_k}{2k\pi i} = \begin{cases} 0 & \text{for even } k \\ \frac{4}{2k\pi i} = -\frac{2i}{k\pi} & \text{for odd } k \end{cases}$$

What about $\hat{f}_0$? Note that $\hat{f}_0 = \langle f, 1 \rangle_{\text{per}}$ is the average of f over a full period of length 1. In our example we have $\hat{f}_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)dx = 0$. Recall that we obtained the same values for $\hat{f}_k$ earlier in (25).

4 Periodic function, discrete time: discrete Fourier transform

4.1 Periodic functions for discrete time values, aliasing

So far we considered 1-periodic functions $f(x)$ defined for $x \in \mathbb{R}$, and we wrote this function as a linear combination of the basis functions $u^{(k)}(x) = e^{2\pi i k x}$ for $k \in \mathbb{Z}$.

Now we consider a step size $1/N$ and the discrete points $x_j = j/N$ for $j \in \mathbb{Z}$. A discrete time signal is given by values f_j at the points x_j, $j \in \mathbb{Z}$. As the signal is periodic we have

$$\boxed{f_{j+N} = f_j \qquad \text{for } j \in \mathbb{Z}}$$

Hence we only need to specify N values. By convention we use the values $f_0, \ldots, f_{N-1}$:

$$\vec{f} = [f_0, \ldots, f_{N-1}]^\top \in \mathbb{C}^N.$$

We now want to use the values of the trigonometric functions $u^{(k)}(x)$ at the points x_j: This function has the values

$$\boxed{\vec{u}^{(k)} := \left[e^{2\pi i k \cdot 0/N}, \ldots, e^{2\pi i k \cdot j/N}\right]^\top}$$

We have however so-called **aliasing**: $\boxed{\vec{u}^{(k+N)} = \vec{u}^{(k)}}$

Proof:

$$u^{(k+N)}(x_j) = e^{2\pi i (k+N) j/N} = e^{2\pi i j k/N} \underbrace{e^{2\pi i j}}_{1} = u^{(k)}(x_j)$$

The functions $u^{(k)}(x)$, $u^{(k+N)}(x)$, $u^{(k+2N)}(x), \ldots$ have all exactly the **same values at the grid points** x_j (on all the other points in $\mathbb{R}$ they are different, though):This means that the vectors $\vec{u}^{(k)}$ for all $k \in \mathbb{Z}$ are actually only N different vectors which keep repeating.

Example: Let e.g. $N = 5$:

$$\ldots, \underbrace{\vec{u}^{(-2)}}_{\vec{u}^{(3)}}, \underbrace{\vec{u}^{(-1)}}_{\vec{u}^{(4)}}, \vec{u}^{(0)}, \vec{u}^{(1)}, \vec{u}^{(2)}, \vec{u}^{(3)}, \vec{u}^{(4)}, \underbrace{\vec{u}^{(5)}}_{\vec{u}^{(0)}}, \underbrace{\vec{u}^{(6)}}_{\vec{u}^{(1)}}, \ldots$$

In order to represent the vector $\vec{f} = [f_0, \ldots, f_4]^\top \in \mathbb{C}^5$ we need 5 basis functions. By convention we use the vectors $\vec{u}^{(0)}, \vec{u}^{(1)}, \vec{u}^{(2)}, \vec{u}^{(3)}, \vec{u}^{(4)}$. We could also have picked the vectors $\vec{u}^{(-2)}, \vec{u}^{(-1)}, \vec{u}^{(0)}, \vec{u}^{(1)}, \vec{u}^{(2)}$, but these are exactly the same vectors, just in a different order.

Given: values of signal in time domain: $\vec{f} = [f_0, f_1, f_2, f_3, f_4]^\top \in \mathbb{C}^5$

Wanted: write $\vec{f}$ as linear combination of $\vec{u}^{(0)}, \ldots, \vec{u}^{(4)}$:

$$\boxed{\vec{f} = \hat{f}_0 \vec{u}^{(0)} + \hat{f}_1 \vec{u}^{(1)} + \hat{f}_2 \vec{u}^{(2)} + \hat{f}_3 \vec{u}^{(3)} + \hat{f}_4 \vec{u}^{(4)}}$$

We want to find the vector $\vec{\hat{f}} = \left[\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_3, \hat{f}_4\right]^\top \in \mathbb{C}^5$ which represents the signal in the frequency domain.

Once we have this vector $\vec{\hat{f}}$ we can also write $\vec{f}$ as a linear combination of the vectors $\underbrace{\vec{u}^{(-2)}}_{\vec{u}^{(3)}}, \underbrace{\vec{u}^{(-1)}}_{\vec{u}^{(4)}}, \vec{u}^{(0)}, \vec{u}^{(1)}, \vec{u}^{(2)}$:

$$\boxed{\vec{f} = \hat{f}_3 \vec{u}^{(-2)} + \hat{f}_4 \vec{u}^{(-1)} + \hat{f}_0 \vec{u}^{(0)} + \hat{f}_1 \vec{u}^{(1)} + \hat{f}_2 \vec{u}^{(2)}}$$

4.2 The discrete Fourier transform $\mathscr{F}_N$

Given: values of signal in time domain: $\vec{f} = [f_0, f_1, \dots, f_{N-1}]^\top \in \mathbb{C}^N$.

Wanted: write $\vec{f}$ as linear combination of $\vec{u}^{(0)}, \dots, \vec{u}^{(N-1)}$:

$$\vec{f} = \hat{f}_0\vec{u}^{(0)} + \hat{f}_1\vec{u}^{(1)} + \cdots + \hat{f}_{N-1}\vec{u}^{(N-1)}$$
$$f_j = \sum_{k=0}^{N-1} \hat{f}_k e^{2\pi i k j/N}$$

We need to find the coefficient vector $\vec{\hat{f}} = [\hat{f}_0, \hat{f}_1, \dots, \hat{f}_{N-1}]$.

We claim that the vectors $\vec{u}^{(0)}, \dots, \vec{u}^{(N-1)}$ form an orthogonal basis of $\mathbb{C}^N$: We have

$$\left(\vec{u}^{(k)}, \vec{u}^{(l)}\right) = \sum_{j=0}^{N-1} e^{2\pi i k j/N} e^{-2\pi i l j/N} = \sum_{j=0}^{N-1} a^j \qquad a := e^{2\pi i (k-l)/N}$$

For $k = l$ we have $a = 1$ and hence

$$\left(\vec{u}^{(k)}, \vec{u}^{(k)}\right) = N.$$

For $k, l \in \{0, \dots, N-1\}$ with $k \neq l$ we have $a \neq 1$ and

$$\left(\vec{u}^{(k)}, \vec{u}^{(l)}\right) = \sum_{j=0}^{N-1} a^j = \frac{a^N - 1}{a-1} = \frac{e^{2\pi i (k-l)} - 1}{a-1} = \frac{1-1}{a-1} = 0.$$

Now we can use (5) to write a vector $\vec{f} = (f_0, \dots, f_{N-1}) \in \mathbb{C}^N$ as a linear combination of the vectors $\vec{u}^{(0)}, \dots, \vec{u}^{(N-1)}$:

$$\vec{f} = \sum_{k=0}^{N-1} \hat{f}_k \vec{u}^{(k)} \qquad \hat{f}_k = \frac{\left(\vec{f}, \vec{u}^{(k)}\right)}{\left(\vec{u}^{(k)}, \vec{u}^{(k)}\right)} = \frac{1}{N}\sum_{j=0}^{N-1} f_j e^{-2\pi i k j/N} \tag{51}$$

Given a vector $\vec{f} = [f_0, \dots, f_{N-1}]^\top \in \mathbb{C}^N$ in the "time domain" we obtain a vector $\vec{\hat{f}} = [\hat{f}_0, \dots, \hat{f}_{N-1}] \in \mathbb{C}^N$ in the frequency domain. This operation is called the **discrete Fourier transform** $\mathscr{F}_N$, its inverse is called $\mathscr{F}_N^{-1}$:

$$\vec{\hat{f}} = \mathscr{F}_N \vec{f}, \qquad \vec{f} = \mathscr{F}_N^{-1} \vec{\hat{f}}$$

4.3 Discrete Fourier transform for $N = 4, 8$

Let $\boxed{\omega := e^{2\pi i/N}}$. This corresponds to a **rotation by an angle of** $2\pi/N$. Then using the basis vectors

$$\vec{u}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \vec{u}^{(1)} = \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \vdots \\ \omega^{N-1} \end{bmatrix}, \quad \vec{u}^{(2)} = \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \vdots \\ \omega^{2(N-1)} \end{bmatrix}, \quad \vec{u}^{(3)} = \begin{bmatrix} 1 \\ \omega^3 \\ \omega^6 \\ \vdots \\ \omega^{3(N-1)} \end{bmatrix}, \quad \dots \quad , \vec{u}^{(N-1)} = \begin{bmatrix} 1 \\ \omega^{N-1} \\ \omega^{2(N-1)} \\ \vdots \\ \omega^{(N-1)(N-1)} \end{bmatrix}$$

we want to write a given $\vec{f} \in \mathbb{C}^N$ as $\vec{f} = \hat{f}_1\vec{u}^{(1)} + \cdots + \hat{f}_{N-1}\vec{u}^{(N-1)}$. We have

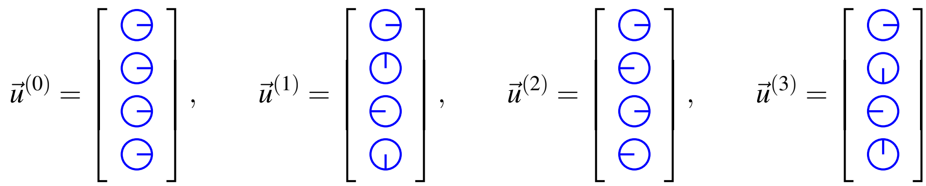
$$f_j = \sum_{k=0}^{N-1} \hat{f}_k \omega^{jk}, \qquad \hat{f}_k = \frac{1}{N}\sum_{j=0}^{N-1} f_j \omega^{-jk}$$

For $\boxed{N = 4}$ we have $\omega = e^{2\pi i/4} = e^{i\pi/2} = i$ which is a rotation by $\frac{\pi}{2}$, i.e., 90°.

Therefore we get the basis vectors

$$\vec{u}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \vec{u}^{(1)} = \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}, \qquad \vec{u}^{(2)} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \qquad \vec{u}^{(3)} = \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix}$$

Imagine a **wheel** which starts in position at time 0, and rotates with a **speed of** k **rotations per second** ($k > 0$ means counterclockwise, $k < 0$ means clockwise). We **take snapshots** N **times per second**: at times $\frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}$ (then it just repeats). Note that due to **aliasing** $\vec{u}^{(3)}$ is the same as $\vec{u}^{(-1)}$. If we show the snapshots of $\vec{u}^{(3)}$ as a movie with $N = 4$ frames per second the wheel would appear to rotate backwards with 1 rotation per second, rather than forward with 3 rotations per second. This is the reason that in movies wheels appear to rotate backwards sometimes.



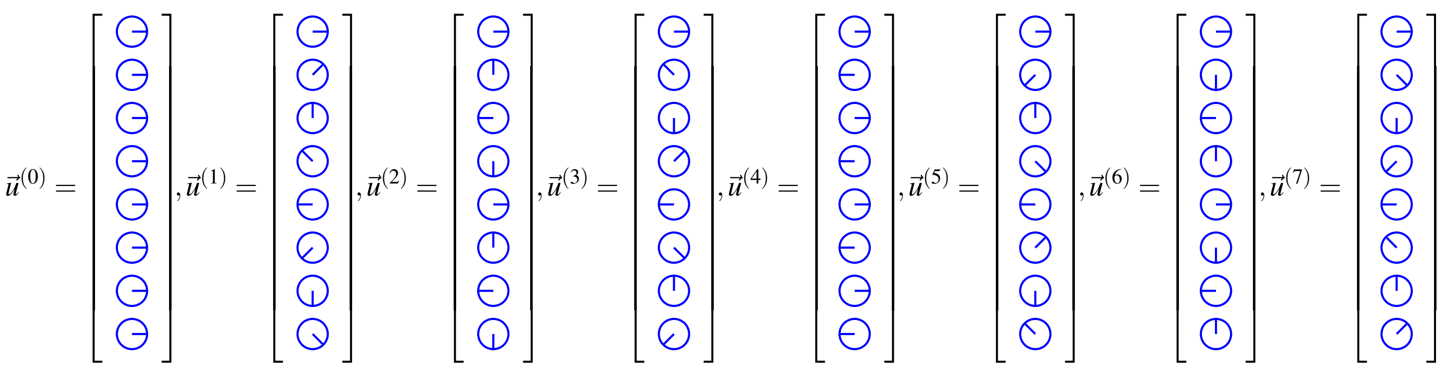
Therefore we can write $\vec{f} = \mathscr{F}_N^{-1}\vec{\hat{f}}$ and $\vec{\hat{f}} = \mathscr{F}_N \vec{f}$ as

$$\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{bmatrix}, \qquad \begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

We get for example

$$\mathscr{F}_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathscr{F}_4 \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} + \frac{1}{2}i \\ -\frac{1}{2} \\ -\frac{1}{2} - \frac{1}{2}i \end{bmatrix}$$

For $\boxed{N = 8}$ we take snapshots 8 times per second at times $\frac{0}{8}, \ldots, \frac{7}{8}$ and obtain the basis vectors



where $\omega = \; = (1+i)/\sqrt{2}$, $\; = (-1+i)/\sqrt{2}$, $\; = (-1-i)/\sqrt{2}$, $\; = (1-i)/\sqrt{2}$. Note that $\vec{u}^{(0)}$ is a fixed wheel. $\vec{u}^{(1)}, \vec{u}^{(2)}, \vec{u}^{(3)}$ are wheels rotating with $1, 2, 3$ rotations per second whereas $\vec{u}^{(7)} = \vec{u}^{(-1)}, \vec{u}^{(6)} = \vec{u}^{(-2)}, \vec{u}^{(5)} = \vec{u}^{(-3)}$ are wheels rotating with $-1, -2, -3$ rotations per second (i.e., clockwise). For $\vec{u}^{(4)} = \vec{u}^{(-4)}$ both interpretations are equally valid.

4.4 Discrete Fourier transform in Matlab

The discrete Fourier transform $\mathscr{F}_N$ maps the vector $\vec{f} = [f_0, \ldots, f_{N-1}]$ in the time domain to the vector $\vec{\hat{f}} = [\hat{f}_0, \ldots, \hat{f}_{N-1}]$ in the frequency domain. The inverse discrete Fourier transform $\mathscr{F}_N^{-1}$ maps the vector $\vec{\hat{f}}$ back to the vector $\vec{f}$:

$$\vec{\hat{f}} = \mathscr{F}_N \vec{f}, \qquad \vec{f} = \mathscr{F}_N^{-1} \vec{\hat{f}}$$

As an example consider the vector $\vec{f} = [1,1,1,1]$ in the time domain. Since this corresponds to the function $\tilde{f}(x) = 1$ the Fourier vector is $\vec{\hat{f}} = [1,0,0,0]$:

$$\mathscr{F}_4[1,1,1,1] = [1,0,0,0], \qquad \mathscr{F}_4^{-1}[1,0,0,0] = [1,1,1,1].$$

We use the m-files **four.m**

```
function fh = four(f)
fh = fft(f)/length(f);
```

and **ifour.m**

```
function f = ifour(fh)
f = ifft(fh)*length(fh);
```

Then we can compute the discrete Fourier transform in Matlab as follows:

```
>> f = [1,1,1,1]
f =      1     1     1     1
>> fh = four(f)
fh =      1     0     0     0
>> ifour(fh)
ans =      1     1     1     1
>> four([1 0 0 0])
ans =
        0.25         0.25         0.25         0.25
>> four([0 1 2 3])
ans =    1.5 + 0i      -0.5 + 0.5i      -0.5 + 0i      -0.5 - 0.5i
```

In Matlab all indices are shifted by 1: You can access components of a Matlab vector `v` using `v(1)`, `v(2)`, ... Note that Matlab begins counting with 1, whereas our signal and Fourier vectors are indexed from 0 to $N-1$. Therefore in Matlab

f_j corresponds to `f(j+1)`

$\hat{f}_k$ corresponds to `fh(k+1)`

In our example we have $\vec{\hat{f}} = \left[\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_3\right] = [1,0,0,0]$ and $\hat{f}_0$ corresponds to `fh(1)` in Matlab.

4.5 Properties of the discrete Fourier transform

Here we summarize the key properties:

$$\boxed{\begin{array}{lcll} \hat{f}_k = \dfrac{1}{N}\displaystyle\sum_{j=0}^{N-1} f_j e^{-2\pi i k j/N} & \Leftrightarrow & f_j = \displaystyle\sum_{k=0}^{N-1} \hat{f}_k e^{2\pi i k j/N} & \textbf{Inversion} \\ \dfrac{1}{N}\displaystyle\sum_{j=0}^{N-1} \left|f_j\right|^2 & = & \displaystyle\sum_{k=0}^{N-1} \left|\hat{f}_k\right|^2 & \textbf{Parseval} \\ q_j = \dfrac{1}{N}\displaystyle\sum_{\ell=0}^{N-1} f_\ell g_{j-\ell} & \Leftrightarrow & \hat{q}_k = \hat{f}_k \hat{g}_k & \textbf{Convolution} \end{array}} \tag{52}$$

Recall that

$$\left(\vec{u}^{(k)}, \vec{u}^{(\ell)}\right) = \begin{cases} N & \text{for } k = l \\ 0 & \text{for } k \neq l \end{cases}$$

This gave us the formula for $\hat{f}_k$ in (51), and (7) gives

$$(\vec{f}, \vec{f}) = \sum_{k=0}^{N-1} \left|\hat{f}_k\right|^2 \underbrace{\left(\vec{u}^{(k)}, \vec{u}^{(k)}\right)}_{N}$$

yielding the Parseval formula. The Parseval formula implies that inner products in the time domain correspond to inner products in the frequency domain: $\frac{1}{N}\left(\vec{f},\vec{g}\right) = \left(\vec{\hat{f}},\vec{\hat{g}}\right)$, i.e.,

$$\boxed{\frac{1}{N}\sum_{j=0}^{N-1} f_j\overline{g_j} = \sum_{k=0}^{N-1}\hat{f}_k\overline{\hat{g}_k}}$$

Note that the sums on the left hand side in (52) correspond to the integrals on the left hand side in (43) if we apply a midpoint rule to the integrals.

For the convolution we plug in the Fourier representation of $g_{j-\ell}$ and switch the order of the sums:

$$q_j = \frac{1}{N}\sum_{\ell=0}^{N-1} f_\ell g_{j-\ell} = \frac{1}{N}\sum_{\ell=0}^{N-1} f_\ell \sum_{k=0}^{N-1}\hat{g}_k e^{2\pi i(j-\ell)k/N}$$

$$q_j = \sum_{k=0}^{N-1}\underbrace{\left(\sum_{\ell=0}^{N-1} f_\ell e^{-2\pi i\ell k/N}\right)}_{\hat{f}_k}\hat{g}_k e^{2\pi ikj/N}$$

Hence we have $q_j = \sum_{k=0}^{N-1}\hat{q}_k e^{2\pi ikj/N}$ with $\hat{q}_k = \hat{f}_k\hat{g}_k$. Remember that the signals are periodic (wrap-around): $g_j = g_{j+N}$. So $g_{j-\ell}$ with $j-\ell < 0$ has to be interpreted in this way.

4.6 Interpretation of $\hat{f}_0,\ldots,\hat{f}_{N-1}$ and the interpolating function $\tilde{f}(x)$

Case of odd N

Assume we have for $N=5$ a vector $\vec{f} = [f_0,\ldots,f_4]$ and we compute the discrete Fourier transform $\vec{\hat{f}} = \left[\hat{f}_0,\hat{f}_1,\hat{f}_2,\hat{f}_3,\hat{f}_4\right]$. Somehow these coefficients $\hat{f}_k$ tell us something about which frequencies are present in our signal. However, it is not quite obvious how this works.

First consider the function

$$\tilde{F}(x) = \hat{f}_0 u^{(0)}(x) + \hat{f}_1 u^{(1)}(x) + \hat{f}_2 u^{(2)}(x) + \hat{f}_3 u^{(3)}(x) + \hat{f}_4 u^{(4)}(x)$$

This function satisfies $\tilde{F}(x_j) = f_j$ for $j = 0,\ldots,N$, i.e., it passes through the points of our discrete signal ("the function $\tilde{F}(x)$ interpolates the given data points"). However, this is not very useful:

- Even if the values $f_0,\ldots,f_4$ are real, the values of $\tilde{F}(x)$ will be complex and non-real for x between the grid points. The problem is that we use only $e^{2\pi ikx}$ for $k \geq 0$, so we cannot get terms like $\cos(2\pi x) = \left(e^{2\pi ix} + e^{-2\pi ix}\right)$.
- We use functions $u^{(k)}$ with frequencies up to $k = 4$. So we use functions with high frequencies to represent data on a fairly coarse grid.

But remember that on the grid points $x_j = j/N$ the function $u^{(3)}$ coincides with $u^{(-2)}$, and the function $u^{(4)}$ coincides with $u^{(-1)}$ (for $x \neq x_j$ these functions are different though). Hence we can consider the function

$$\tilde{f}(x) = \hat{f}_0 u^{(0)}(x) + \hat{f}_1 u^{(1)}(x) + \hat{f}_2 u^{(2)}(x) + \hat{f}_3 u^{(-2)}(x) + \hat{f}_4 u^{(-1)}(x)$$

which still passes through the given data points. Now we have $k = -2,-1,0,1,2$ and therefore $\tilde{f} \in \mathscr{T}_2$. Therefore we can write $\tilde{f}$ in the form

$$\tilde{f}(x) = a_0 + a_1\cos(2\pi x) + b_1\sin(2\pi x) + a_2\cos(2\pi\cdot 2x) + b_2\sin(2\pi\cdot 2x).$$

This function $\tilde{f}(x)$ is much nicer than $\tilde{F}(x)$:

- If the values $f_0,\ldots,f_4$ are real, the values of $\tilde{f}(x)$ will be real for all x.
- We use a function in $\mathscr{T}_2$ with frequencies only up to $k = 2$.

For general odd $N = 2n+1$ the Fourier vector $\left[\hat{f}_0, \hat{f}_1, \dots, \hat{f}_{N-1}\right]$ yields the interpolating function $\tilde{f} \in \mathscr{T}_n$ given by

$$\tilde{f}(x) = \hat{f}_0 u^{(0)}(x) + \hat{f}_1 u^{(1)}(x) + \cdots + \hat{f}_n u^{(n)}(x) + \hat{f}_{N-n} u^{(-n)}(x) + \cdots + \hat{f}_{N-1} u^{(-1)}(x) \tag{53}$$

Therefore the entries of the Fourier vector should be interpreted as follows:

frequency	0	1	2	$\cdots$	n	n	$\cdots$	2	1
	$\hat{f}_0$	$\hat{f}_1$	$\hat{f}_2$	$\cdots$	$\hat{f}_n$	$\hat{f}_{n+1}$	$\cdots$	$\hat{f}_{N-2}$	$\hat{f}_{N-1}$

So $\hat{f}_0$ corresponds to frequency 0 with $u^{(0)}(x) = 1$, it gives the mean value of our data. Then $\hat{f}_1$ and $\hat{f}_{N-1}$ together tell us about frequency 1. The highest frequency terms $\hat{f}_n, \hat{f}_{n+1}$ for frequency n are in the middle of the vector.

Case of even N

Consider for $N = 6$ a discrete signal a vector $\vec{f} = [f_0, \dots, f_5]$ with the discrete Fourier transform $\vec{\hat{f}} = \left[\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_3, \hat{f}_4, \hat{f}_5\right]$. Here the "bad" function $\tilde{F}(x)$ would be

$$\tilde{F}(x) = \hat{f}_0 u^{(0)}(x) + \hat{f}_1 u^{(1)}(x) + \hat{f}_2 u^{(2)}(x) + \hat{f}_3 u^{(3)}(x) + \hat{f}_4 u^{(4)}(x) + \hat{f}_5 u^{(5)}(x).$$

On the grid points x_j the function $u^{(5)}$ coincides with $u^{(-1)}$, the function $u^{(4)}$ coincides with $u^{(-2)}$ and the function $u^{(3)}$ coincides with $u^{(-3)}(x)$. So we replace $\hat{f}_4 u^{(4)}(x) + \hat{f}_5 u^{(5)}(x)$ with $\hat{f}_4 u^{(-2)}(x) + \hat{f}_5 u^{(-1)}(x)$. What should we do with $\hat{f}_3 u^{(3)}(x)$? If we leave it as it is, or if we replace it with $\hat{f}_3 u^{(-3)}(x)$ we would get something "unbalanced", causing a complex function for real data. The answer is to replace $\hat{f}_3 u^{(3)}(x)$ with $\hat{f}_3 \frac{1}{2}\left[u^{(3)}(x) + u^{(-3)}(x)\right]$, yielding the function

$$\tilde{f}(x) = \hat{f}_0 u^{(0)}(x) + \hat{f}_1 u^{(1)}(x) + \hat{f}_2 u^{(2)}(x) + \hat{f}_3 \tfrac{1}{2}\left[u^{(3)}(x) + u^{(-3)}(x)\right] + \hat{f}_4 u^{(-2)}(x) + \hat{f}_5 u^{(-1)}(x)$$

which still passes through the given data points. Note that

$$\tfrac{1}{2}\left[u^{(3)}(x) + u^{(-3)}(x)\right] = \tfrac{1}{2}\left[e^{2\pi i \cdot 3x} + e^{-2\pi i \cdot 3x}\right] = \cos(2\pi \cdot 3x)$$

so that we can write $\tilde{f}$ in the form

$$\tilde{f}(x) = a_0 + a_1 \cos(2\pi x) + b_1 \sin(2\pi x) + a_2 \cos(2\pi \cdot 2x) + b_2 \sin(2\pi \cdot 2x) + a_3 \cos(2\pi \cdot 3x)$$

We denote these functions by $\mathscr{T}_{3,\cos}$ since we take the functions in $\mathscr{T}_3$, but only use the cosine term for frequency 3. Note that it does not make sense to use $\sin(2\pi \cdot 3x)$ since this function is zero at all grid points $x_j = j/6$.

$$\boxed{\mathscr{T}_{n,\cos} := \operatorname{span}\{\underbrace{1}_{\text{freq } 0}, \underbrace{\cos(2\pi x), \sin(2\pi x)}_{\text{freq } 1}, \dots, \underbrace{\cos(2\pi(n-1)x), \sin(2\pi(n-1)x)}_{\text{freq } n-1}, \underbrace{\cos(2\pi n x)}_{\text{freq } n}\}}$$

For general even $N = 2n$ the Fourier vector $(\hat{f}_0, \hat{f}_1, \dots, \hat{f}_{N-1})$ yields the interpolating function $\tilde{f} \in \mathscr{T}_{n,\cos}$ given by

$$\tilde{f}(x) = \hat{f}_0 u^{(0)}(x) + \hat{f}_1 u^{(1)}(x) + \cdots + \hat{f}_{n-1} u^{(n-1)}(x) + \hat{f}_n \frac{1}{2}\left(u^{(n)}(x) + u^{(-n)}(x)\right) + \hat{f}_{N-n+1} u^{(-n+1)}(x) + \cdots + \hat{f}_{N-1} u^{(-1)}(x) \tag{54}$$

Therefore the entries of the Fourier vector should be interpreted as follows:

frequency	0	1	2	$\cdots$	$n-1$	n	$n-1$	$\cdots$	2	1
	$\hat{f}_0$	$\hat{f}_1$	$\hat{f}_2$	$\cdots$	$\hat{f}_{n-1}$	$\hat{f}_n$	$\hat{f}_{n+1}$	$\cdots$	$\hat{f}_{N-2}$	$\hat{f}_{N-1}$

Summary

Let us define the N dimensional space V_N of trigonometric polynomials:

$$V_N := \begin{cases} \mathscr{T}_n & \text{if } N = 2n+1 \text{ is odd} \\ \mathscr{T}_{n,\cos} & \text{if } N = 2n \text{ is even} \end{cases}$$

Then we obtain

Proposition 6. *For given samples f_j at $x_j = j/N$ for $j = 0,\ldots,N-1$ there is a* ***unique interpolating function*** *$\tilde{f} \in V_N$. We can find $\tilde{f}$ using $\vec{\hat{f}} := \mathscr{F}_N \vec{f}$ and*

$$\tilde{f} := \begin{cases} \hat{f}_0 u^{(0)} + \hat{f}_1 u^{(1)} + \cdots + \hat{f}_n u^{(n)} + \hat{f}_{n+1} u^{(-n)} + \cdots + \hat{f}_{N-1} u^{(-1)} & N = 2n+1 \ \textit{odd} \\ \hat{f}_0 u^{(0)} + \hat{f}_1 u^{(1)} + \cdots + \hat{f}_{n-1} u^{(n-1)} + \hat{f}_n \frac{1}{2}\left[u^{(n)} + u^{(-n)}\right] + \hat{f}_{n+1} u^{(-n+1)} + \cdots + \hat{f}_{N-1} u^{(-1)} & N = 2n \ \textit{even} \end{cases} \tag{55}$$

Proof. Obviously $\tilde{f} \in V_N$. Evaluating $\tilde{f}$ at $\frac{0}{N},\ldots,\frac{N-1}{N}$ gives for each term $u^{(k)}$ the vector $\vec{u}^{(k)}$. For terms with $k < 0$ we use aliasing $\vec{u}^{(k)} = \vec{u}^{(k+N)}$, and obtain $\left[\tilde{f}(\frac{0}{N}),\ldots,\tilde{f}(\frac{N-1}{N})\right] = \hat{f}_0 \vec{u}^{(0)} + \cdots + \hat{f}_{N-1} \vec{u}^{(N-1)} = \vec{f}$. We have a one-to-one correspondence of Fourier vectors $\vec{\hat{f}} \in \mathbb{C}^N$ and functions in V_N:

$$\begin{array}{ccccc} \mathbb{C}^N & \overset{\mathscr{F}_N}{\to} & \mathbb{C}^N & \leftrightarrow & V_N \\ \vec{f} & \overset{\mathscr{F}_N}{\to} & \vec{\hat{f}} & \leftrightarrow & \tilde{f} \end{array}$$

Since the mapping $\mathscr{F}_N \colon \mathbb{C}^N \to \mathbb{C}^N$ is also one-to-one we obtain: For any given vector $\vec{f} \in \mathbb{C}^N$ in the time domain there exists a unique interpolating function $\tilde{f} \in V_N$. □

A Fourier vector $\left[\hat{f}_0,\ldots,\hat{f}_{N-1}\right]$ stands for a function $\tilde{f} \in V_N$ given by the following linear combinations of $u^{(-n)},\ldots,u^{(n)}$:

odd $N = 2n+1$: $\tilde{f} \in \mathscr{T}_n$

$\hat{f}_0$	$\hat{f}_1$	$\hat{f}_2$	$\cdots$	$\hat{f}_n$	$\hat{f}_{n+1}$	$\cdots$	$\hat{f}_{N-2}$	$\hat{f}_{N-1}$
$u^{(0)}$	$u^{(1)}$	$u^{(2)}$	$\cdots$	$u^{(n)}$	$u^{(-n)}$	$\cdots$	$u^{(-2)}$	$u^{(-1)}$

even $N = 2n$: $\tilde{f} \in \mathscr{T}_{n,\cos}$

$\hat{f}_0$	$\hat{f}_1$	$\hat{f}_2$	$\cdots$	$\hat{f}_{n-1}$	$\hat{f}_n$	$\hat{f}_{n+1}$	$\cdots$	$\hat{f}_{N-2}$	$\hat{f}_{N-1}$
$u^{(0)}$	$u^{(1)}$	$u^{(2)}$	$\cdots$	$u^{(n-1)}$	$\frac{1}{2}[u^{(n)} + u^{(-n)}]$	$u^{(-n+1)}$	$\cdots$	$u^{(-2)}$	$u^{(-1)}$

4.7 "Upsampling": Evaluating $\tilde{f}(x)$ on a finer grid

Recall that a Fourier vector $\vec{\hat{f}} = \left\|\hat{f}_0,\ldots,\hat{f}_{N-1}\right\|$ corresponds to a function $\tilde{f} \in V_N$:

- $\tilde{f} \in \mathscr{T}_n$ given by (53) for odd $N = 2n+1$
- $\tilde{f} \in \mathscr{T}_{n-1,\cos}$ given by (54) for even $N = 2n$.

Applying $\mathscr{F}_N^{-1}$ to $\vec{\hat{f}}$ gives back the values $f_0,\ldots,f_{N-1}$ on the grid points $x_j = j/N$.

Often we would like to **evaluate the function** $\tilde{f}(x)$ **on a finer grid** $X_j = j/M$, $j = 0,\ldots,M-1$ with $M > N$. We can do this in the following way: E.g., for a vector $\vec{\hat{f}} = \left[\hat{f}_0,\hat{f}_1,\hat{f}_2,\hat{f}_3,\hat{f}_4\right]$ of length $N = 5$ we have the function

$$\tilde{f}(x) = \hat{f}_3 u^{(-2)}(x) + \hat{f}_4(x) u^{(-1)}(x) + \hat{f}_0 u^{(0)}(x) + \hat{f}_1 u^{(1)}(x) + \hat{f}_2(x) u^{(2)}(x) \in \mathscr{T}_2$$

The function does not change if we add zero terms with $u^{(-3)}(x)$ and $u^{(3)}(x)$:

$$\tilde{f}(x) = \underbrace{0}_{\hat{g}_4} \cdot u^{(-3)}(x) + \underbrace{\hat{f}_3}_{\hat{g}_5} u^{(-2)}(x) + \underbrace{\hat{f}_4}_{\hat{g}_6} u^{(-1)}(x) + \underbrace{\hat{f}_0}_{\hat{g}_0} u^{(0)}(x) + \underbrace{\hat{f}_1}_{\hat{g}_1} u^{(1)}(x) + \underbrace{\hat{f}_2}_{\hat{g}_2} u^{(2)}(x) + \underbrace{0}_{\hat{g}_3} \cdot u^{(3)}(x)$$

which corresponds to the Fourier vector

$$\vec{\hat{g}} = \left[\hat{f}_0, \hat{f}_1, \hat{f}_2, 0, 0, \hat{f}_3, \hat{f}_4\right]$$

which is obtained from $\vec{\hat{f}}$ by "sticking zeros into the middle of the vector". Now applying the inverse Fourier transform

$$\vec{g} := \mathscr{F}_7^{-1}\vec{\hat{g}}$$

gives the values $g_j = \tilde{f}(j/M)$ of the function $\tilde{f}$ on the finer grid.

If we start instead with a vector $\vec{\hat{f}} = \left[\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_3\right]$ of length $N = 4$ we obtain the interpolating function

$$\tilde{f}(x) = \tfrac{1}{2}\hat{f}_2 u^{(-2)}(x) + \hat{f}_3(x)u^{(-1)}(x) + \hat{f}_0 u^{(0)}(x) + \hat{f}_1 u^{(1)}(x) + \tfrac{1}{2}\hat{f}_2(x)u^{(2)}(x) \in \mathscr{T}_{2,\cos}$$

therefore the vector $\vec{\hat{g}}$ of length $M = 7$ which corresponds to the same interpolating function is

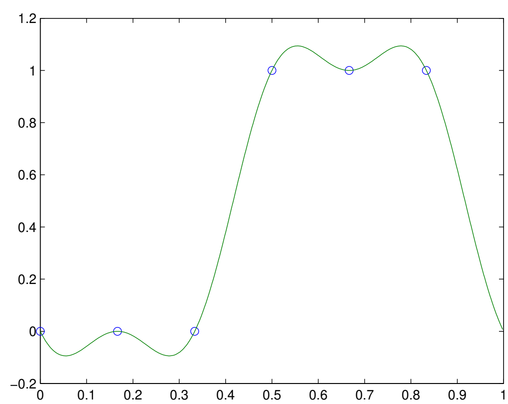
$$\vec{\hat{g}} = \left[\hat{f}_0, \hat{f}_1, \tfrac{1}{2}\hat{f}_2, 0, 0, \tfrac{1}{2}\hat{f}_2, \hat{f}_3\right]$$

Summary: A vector $\vec{\hat{f}} = \left[\hat{f}_0, \ldots, \hat{f}_{N-1}\right]$ corresponds to an interpolating function $\tilde{f}(x) \in \begin{cases} \mathscr{T}_n & \text{if } N = 2n+1 \text{ odd} \\ \mathscr{T}_{n,\cos} & \text{if } N = 2n \text{ even} \end{cases}$. A vector $\vec{\hat{g}} = (\hat{g}_0, \ldots, \hat{g}_{M-1})$ of length $M > N$ which corresponds to the same interpolating function is given by

$$\boxed{\vec{\hat{g}} = \begin{cases} \left[\hat{f}_0, \ldots, \hat{f}_n, 0, \ldots, 0, \hat{f}_{n+1}, \hat{f}_{N-1}\right] & \text{if } N = 2n+1 \text{ odd} \\ \left[\hat{f}_0, \ldots, \hat{f}_{n-1}, \frac{1}{2}\hat{f}_n, 0, \ldots, 0, \frac{1}{2}\hat{f}_n, \hat{f}_{n+1}, \ldots, \hat{f}_{N-1}f\right] & \text{if } N = 2n \text{ even} \end{cases}}$$

This is implemented in the **Matlab function `fext`** (see m-file on course web page). We can use this to plot the interpolating function $\tilde{f}$ on a finer grid with M points:

```
x = (0:5)/6;          % grid of given signal
X = (0:511)/512;      % fine grid for plotting
f = [0 0 0 1 1 1];    % given signal
fh = four(f);
gh = fext(fh,512);    % extend to length 512
g = ifour(gh);
plot(x,f,'o',X,g)
```



For $M > N$ we have $V_N \subset V_M$, this corresponds to the function `fext` for the Fourier vectors. The corresponding operation in the time domain is called "**upsampling**":

$$\begin{array}{ccccc} & \mathbb{C}^N & \overset{\mathscr{F}_N}{\to} & \mathbb{C}^N & \leftrightarrow & V_N \\ \text{upsampling} \downarrow & & & \downarrow \texttt{fext} & & \cap \\ & \mathbb{C}^M & \overset{\mathscr{F}_M}{\to} & \mathbb{C}^M & \leftrightarrow & V_M \end{array}$$

4.8 "Low-pass filter": Least squares approximation with lower frequency functions

We are given the values f_j at the points $x_j = \frac{j}{N}$ for $j = 0, \ldots, N-1$. We have seen that we can obtain an interpolating function

$$\tilde{f} \in V_N = \begin{cases} \mathscr{T}_n & \text{for odd } N = 2n+1 \\ \mathscr{T}_{n1,\cos} & \text{for even } N = 2n \end{cases}$$

If we use a space $\mathscr{T}_m$ or $\mathscr{T}_{m,\cos}$ with $m < n$ we cannot expect to find a function $\tilde{g}(x)$ which passes through all points. The best we can do is to minimize the least squares error

$$\sum_{j=0}^{N-1} \left|\tilde{g}(x_j) - f_j\right|^2. \tag{56}$$

Because of the orthogonality of the vectors $\vec{u}^{(k)}$ (see section 4.2) we can obtain the least squares approximation by using only terms which correspond to our subspace (see section 2.3): We have a vector $\vec{f} \in \mathbb{C}^N$ and using the discrete Fourier transform we can write it as

$$\vec{f} = \sum_{k=0}^{N-1} \hat{f}_k \vec{u}^{(k)}$$

Case of odd $M = 2m+1$: We want to approximate the values f_j using functions $\mathscr{T}_m$ with $M = 2m+1 < N$. For example consider $\mathscr{T}_2$ where $m = 2$ and $M = 2m+1 = 5$. In this case we want to use the frequencies $-2,-1,0,1,2$. Because of aliasing the corresponding vectors are $\vec{u}^{(N-2)}, \vec{u}^{(N-1)}, \vec{u}^{(0)}, \vec{u}^{(1)}, \vec{u}^{(2)}$. The best approximation $\vec{G} \in \operatorname{span}\left\{\vec{u}^{(N-2)}, \vec{u}^{(N-1)}, \vec{u}^{(0)}, \vec{u}^{(1)}, \vec{u}^{(2)}\right\}$ is obtained as

$$\vec{G} = \hat{f}_{N-2}\vec{u}^{(N-2)} + \hat{f}_{N-1}\vec{u}^{(N-1)} + \hat{f}_0\vec{u}^{(0)} + \hat{f}_1\vec{u}^{(1)} + \hat{f}_2\vec{u}^{(2)}$$

and corresponds to the Fourier vector $\vec{\hat{G}}$ given by

$$\left[\hat{G}_0, \hat{G}_1, \ldots, \hat{G}_{N-1}\right] = \left[\hat{f}_0, \hat{f}_1, \hat{f}_2, 0, \ldots, 0, \hat{f}_{N-2}, \hat{f}_{N-1}\right]$$

where we replace high frequency coefficients with zeros. The function $\tilde{g}(x)$ corresponds to the Fourier vector $\vec{\hat{g}}$ of length $M = 2m+1 = 5$ given by

$$[\hat{g}_0, \hat{g}_1, \hat{g}_2, \hat{g}_3, \hat{g}_4] = \left[\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_{N-2}, \hat{f}_{N-1}\right]$$

where we "remove the high frequency coefficients from the middle" of the vector $\vec{\hat{f}}$.

To approximate the data values using $\mathscr{T}_m$ with $M = 2m+1 < N$ we use

$$[\hat{g}_0, \ldots, \hat{g}_{M-1}] := \left[\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_m, \hat{f}_{N-m}, \ldots, \hat{f}_{N-1}\right]$$

Case of even $M = 2m$: We want to approximate the values f_j using functions $\mathscr{T}_{m,\cos}$ with $M = 2m < N$. Here we use for the highest frequency m only the cosine term. For example consider $\mathscr{T}_{2,\cos} = \operatorname{span}\{1, \cos(2\pi x), \sin(2\pi x), \cos(4\pi x)\}$ where $m = 2$ and $M = 2m = 4$. The best approximation corresponds to the Fourier vector $\vec{\hat{G}}$ given by

$$\left[\hat{G}_0, \hat{G}_1, \ldots, \hat{G}_{N-1}\right] = \left[\hat{f}_0, \hat{f}_1, \frac{\hat{f}_2 + \hat{f}_{N-2}}{2}, 0, \ldots, 0, \frac{\hat{f}_2 + \hat{f}_{N-2}}{2}, \hat{f}_{N-1}\right].$$

The function $\tilde{g}(x)$ corresponds to the Fourier vector $\vec{\hat{g}}$ of length $M = 2m = 4$ given by

$$[\hat{g}_0, \hat{g}_1, \hat{g}_2, \hat{g}_3] = \left[\hat{f}_0, \hat{f}_1, \hat{f}_2 + \hat{f}_{N-2}, \hat{f}_{N-1}\right].$$

To approximate the data values using $\mathscr{T}_{m,\cos}$ with $M = 2m < N$ we use

$$[\hat{g}_0, \ldots, \hat{g}_{M-1}] := \left[\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_{m-1}, \hat{f}_m + \hat{f}_{N-m}, \hat{f}_{N-m+1}, \ldots, \hat{f}_{N-1}\right]$$

Summary: We can approximate a data vector $[f_0, \ldots, f_{N-1}]$ of length N using a trigonometric space of dimension $M < N$ as follows: define the vector $\vec{g}$ of length M by

$$[\hat{g}_0, \ldots, \hat{g}_{M-1}] := \begin{cases} \left[\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_m, \hat{f}_{N-m}, \ldots, \hat{f}_{N-1}\right] & \text{for odd } M = 2m+1 \\ \left[\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_{m-1}, \hat{f}_m + \hat{f}_{N-m}, \hat{f}_{N-m+1}, \ldots, \hat{f}_{N-1}\right] & \text{for even } M = 2m \end{cases}$$

Then the corresponding function

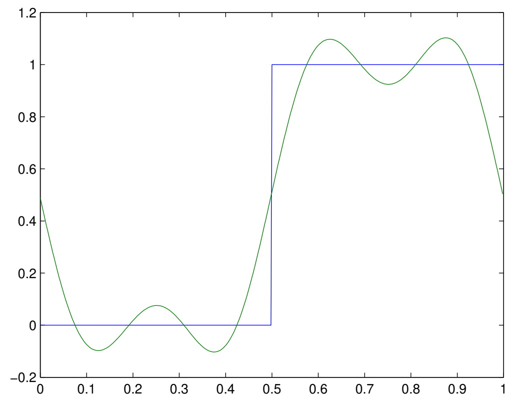
$$\tilde{g}(x) \in \begin{cases} \mathcal{T}_m & \text{for odd } M = 2m+1 \\ \mathcal{T}_{m,\cos} & \text{for even } M = 2m \end{cases}$$

is the best least squares approximation, i.e., it minizes (56).

This operation is implemented in the Matlab function **`ftrunc`** (see m-file on course web page).

Example: We consider a discrete signal which has value 0 for $x \in [0, \frac{1}{2})$ and value 1 for $x \in [\frac{1}{2}, 1)$. We want to find the best least squares approximation using a function $\tilde{g} \in \mathcal{T}_4$, i.e., $M = 9$.

```
x = (0:511)/512;   % grid
f = (x>=.5);       % signal: 1 for x>=.5, 0 else
fh = four(f);
gh = ftrunc(fh,9); % truncate with M=9
Gh = fext(gh,512); % extend for plotting
G = ifour(Gh);
plot(x,f,x,G)
```



We are given a discrete signal $\vec{f} \in \mathbb{C}^N$ in the time domain. We want to apply a **"low-pass filter"**. This means: Find the best least squares approximation by a discrete signal $\vec{G}$ which corresponds to a function $\tilde{g} \in V_M$ with $M < N$. "Best least square approximation" means that $\left\| \vec{f} - \vec{G} \right\|$ is minimal.

$$\begin{array}{cccccc} & \vec{f} \in \mathbb{C}^N & \stackrel{\mathcal{F}_N}{\rightarrow} & \mathbb{C}^N & \leftrightarrow & V_N \\ & & & \downarrow \texttt{ftrunc} & & \downarrow \text{orth. projection} \\ \text{lowpass filter} \downarrow & & & \mathbb{C}^M & \leftrightarrow & V_M \\ & & & \downarrow \texttt{fext} & & \cap \\ & \vec{G} \in \mathbb{C}^N & \stackrel{\mathcal{F}_N}{\rightarrow} & \mathbb{C}^N & \leftrightarrow & V_N \end{array}$$

4.9 Calculus review: absolute convergence

Example: Consider the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. This is an alternating series $\sum_{k=1}^{\infty} a_k$ with $|a_{k+1}| < |a_k|$ and $a_k \to 0$ as $k \to \infty$. By the alternating series theorem this series converges, and one can show

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} \cdots = \ln 2$$

We now reorder the terms in this series: we take two positive terms, one negative term, two positive terms, ... Despite the fact that we add the same terms as before (just in a different order) we now get a different value for the sum

$$\tilde{S} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots = \frac{3}{2} \ln 2$$

The problem is that the **series is not absolutely convergent**: $\sum_{k=1}^{\infty} |a_k| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty$.

A series $\sum_{k=1} a_k$ is called **absolutely convergent** if $\boxed{\sum_{k=1}^{\infty} |a_k| < \infty}$.

In this case we have

- **the series is convergent**
- **for any reordering we obtain the same sum**

4.10 Sampling a function

Sampling a function F at points j/N

We can characterize a 1-periodic function $F(x)$ by its Fourier coefficients $\hat{F}_k$, $k \in \mathbb{Z}$.

Assume that $F \in PW^1$ **without jumps** (each piece has continuous derivative, the function may have kinks). Then we showed that the Fourier series $\sum_{k=-\infty}^{\infty} \hat{F}_k e^{2\pi ikx}$ converges absolutely, i.e., $\sum_{k=-\infty}^{\infty} \left|\hat{F}_k\right| < \infty$.

Evaluating this function on grid points $x_j = j/N$, $j = 0, \ldots, N-1$ gives values

$$f_j = F(j/N)$$

with $f_{j+N} = f_j$. This process which takes us from the function $F(x)$ to the vector $\vec{f} = (f_0, \ldots, f_{N-1})$ is called **sampling**. Obviously, information about the behavior of f between the grid points gets lost during sampling.

By applying the discrete Fourier transform $\mathscr{F}_N$ to $\vec{f}$ we obtain the Fourier coefficients $\left(\hat{f}_0, \ldots, \hat{f}_{N-1}\right)$.

How are Fourier coefficients $\left(\hat{f}_0, \ldots, \hat{f}_{N-1}\right)$ related to the original Fourier coefficients $\hat{F}_k$, $k \in \mathbb{Z}$?

Note that we have $F(x) = \sum_{k=-\infty}^{\infty} \hat{F}_k u^{(k)}(x)$ and hence for $x_j = j/N$

$$F(x_j) = \sum_{k=-\infty}^{\infty} \hat{F}_k u^{(k)}(x_j) \tag{57}$$

Now remember the "aliasing property" of $u^{(k)}(x)$: On the grid points $x_j = j/N$ we have

$$\cdots = u^{(k-N)}(x_j) = u^{(k)}(x_j) = u^{(k+N)}(x_j) = u^{(k+2N)}(x_j) = \cdots$$

Therefore we can simplify (57) by collecting the identical terms together:

$$\underbrace{F(x_j)}_{f_j} = \sum_{k=0}^{N-1} \underbrace{\left(\sum_{\ell=-\infty}^{\infty} \hat{F}_{k+\ell N} \right)}_{\hat{f}_k} u^{(k)}(x_j)$$

Note that we have now reordered the terms in the sum. Since we have absolute convergence $\sum_{k=-\infty}^{\infty} \left|\hat{F}_k\right| < \infty$ the sum after reordering is still $F(x_j)$.

We see that we obtain the discrete Fourier representation $f_j = \sum_{k=0}^{N-1} \hat{f}_k u^{(k)}(x_j)$. We have shown

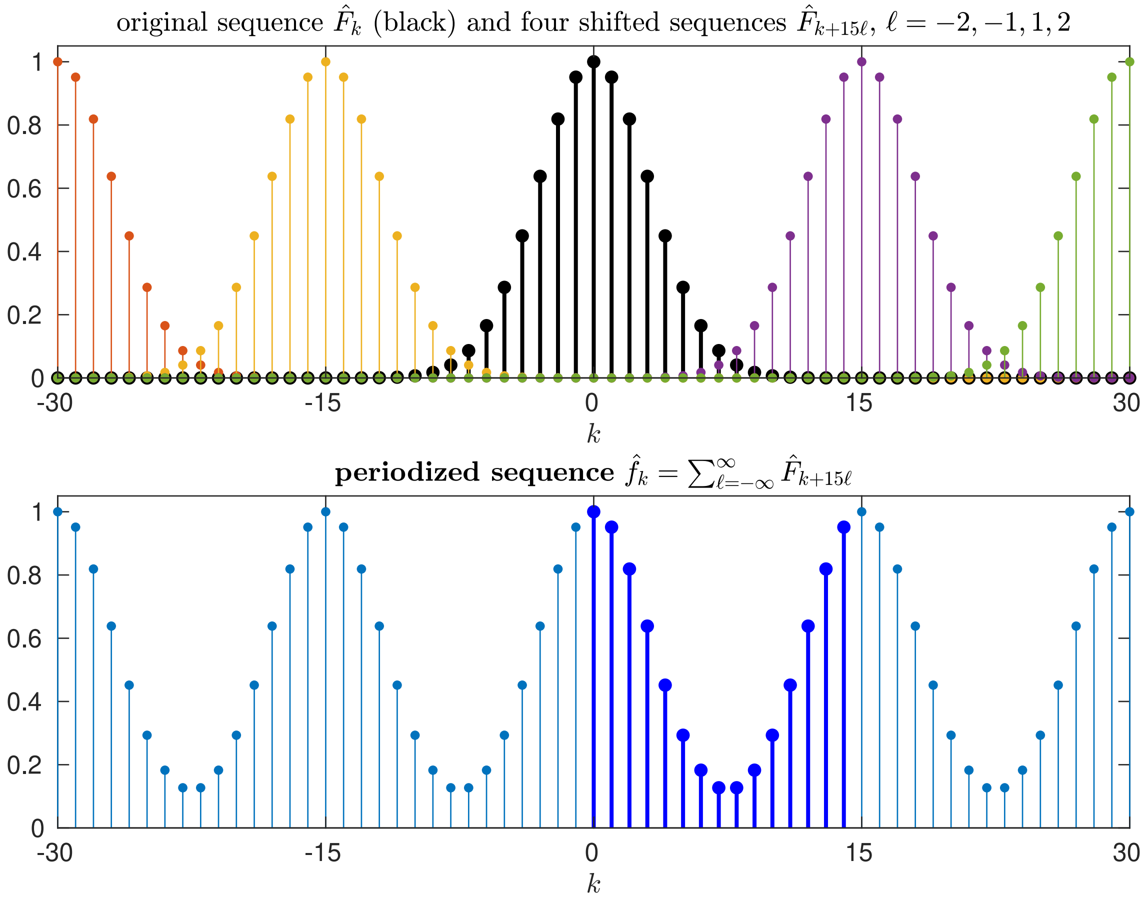
$$\boxed{\hat{f}_k = \sum_{\ell=-\infty}^{\infty} \hat{F}_{k+\ell N}} \tag{58}$$

This is how we get from the coefficients $\hat{F}_k$, $k \in \mathbb{Z}$ of the original function to the coeffcients $\hat{f}_0, \ldots, \hat{f}_{N-1}$ of the sampled version.

Note: Since $\sum_{k=-\infty}^{\infty} \left|\hat{F}_k\right| < \infty$ we have $\sum_{\ell=-\infty}^{\infty} \left|\hat{F}_{k+\ell N}\right| < \infty$. I.e., the series (58) is absolutely convergent and therefore convergent.

This operation in the frequency domain is called "**periodization**":

- start with the sequence of Fourier coefficients $\left(\hat{F}_k\right)_{k\in\mathbb{Z}}$ which decay to 0 for $k \to \pm\infty$
- take the shifted sequences $\left(\hat{F}_{k+\ell N}\right)_{k\in\mathbb{Z}}$ for $\ell = \dots, -2, -1, 0, 1, 2, \dots$ and add them together
- this yields a periodic sequence $\hat{f}_k = \sum_{\ell=-\infty}^{\infty} \hat{F}_{k+\ell N}$ with $\hat{f}_{k+N} = \hat{f}_k$, so we only need to specify $\vec{\hat{f}} = \left[\hat{f}_0, \dots, \hat{f}_{N-1}\right]$



Conclusion: Sampling in time domain corresponds to **periodization in frequency domain**.

4.11 Sampling theorem, Nyquist frequency

When we sample a 1-periodic function $F(x)$ at the grid points $x_j = \frac{j}{N}$ we obtain the discrete values $f_j := F(j/N)$. The Fourier coefficients $\left(\hat{F}_k\right)_{k\in\mathbb{Z}}$ and $\left(\hat{f}_k\right)_{k=0,\dots,N}$ are related by (58). If we only have the sampled values f_j we cannot hope to reconstruct the function $F(x)$: we don't know what happens between the points $x_j = j/N$. For the Fourier coefficients we only know the sums

$$\hat{f}_0 = \cdots + \hat{F}_{-N} + \hat{F}_0 + \hat{F}_N + \hat{F}_{2N} + \cdots, \qquad \hat{f}_1 = \cdots + \hat{F}_{-N+1} + \hat{F}_1 + \hat{F}_{N+1} + \hat{F}_{2N+1} + \cdots, \qquad \text{etc.}$$

so we cannot hope to recover $\hat{F}_k$.

Assume that the function $F(x)$ is **band limited** in the sense that $\boxed{F \in V_N}$, and we are given the samples $f_j := F(\frac{j}{N})$, $j = 0, \dots, N-1$.

How to recover the function F from the samples f_j: Let $\vec{\hat{f}} := \mathscr{F}_N \vec{f}$ and define the interpolating function $\tilde{f} \in V_N$ by (55). By Proposition 6 the interpolating function in V_N is unique, hence $\boxed{F = \tilde{f}}$.

On the other hand, if we only know that $F \in V_{N+1}$ we cannot reconstruct the function F from the samples $F(\frac{j}{N})$, $j = 0, \dots, N-1$: There is a nonzero function $F \in V_{N+1}$ which is zero at all points j/N:

For $N=2n$ **even** let $F(x)=\sin(2\pi nx)\in\mathscr{T}_n=V_{2n+1}$. Then $F(j/N)=\sin\left(2\pi n\frac{j}{2n}\right)=\sin(\pi j)=0$.

For $N=2n+1$ **odd** let $F(x)=\cos(2\pi nx)-\cos(2\pi(n+1)x)$. Then $2\cos(2\pi nx)=u^{(-n)}+u^{(n)}$ has the samples

$$\vec{u}^{(-n)}+\vec{u}^{(n)}\overset{\text{aliasing}}{=}\vec{u}^{(-n+N)}+\vec{u}^{(n-N)}=\vec{u}^{(n+1)}+\vec{u}^{(-n-1)}$$

which are the samples of the function $u^{(n+1)}+u^{(-n-1)}=2\cos(2\pi(n+1)x)$.

This can be formulated as the **sampling theorem**:

Proposition 7. *Assume that $F\in V_M$ and we are given the samples $F(\frac{j}{N})$, $j=0,\dots,N-1$. The function F can by uniquely determined from these samples if and only if $N\geq M$.*

For odd $M=2m+1$ this means:

Proposition 8. *Assume that $F\in\mathscr{T}_m$ and we are given the samples $F(\frac{j}{N})$, $j=0,\dots,N-1$. The function F can by uniquely determined from these samples if and only if $N\geq 2m+1$.*

Note: Sampling with sampling frequency $2m$ is **not** sufficient since $\sin(2\pi mx)$ is zero at the points $x=\frac{j}{2m}$.

If we have samples with sampling frequency N (i.e., spacing $h=\frac{1}{N}$) we can resolve signal frequencies less than $\frac{N}{2}=\frac{1}{2h}$. This critical frequency $\frac{N}{2}=\frac{1}{2h}$ is called "**Nyquist frequency**".

If the condition is satisfied we have

$$F(x)=\tilde{f}(x)$$

where $\tilde{f}\in V_N$ is the interpolating function for the samples f_j.

We have an **explicit formula for** $\tilde{f}$ in terms of the values f_j:

Let $N=2n+1$ **odd** and we are given the samples $f_j=F(\frac{j}{N})$, $j=0,\dots,N-1$. Then

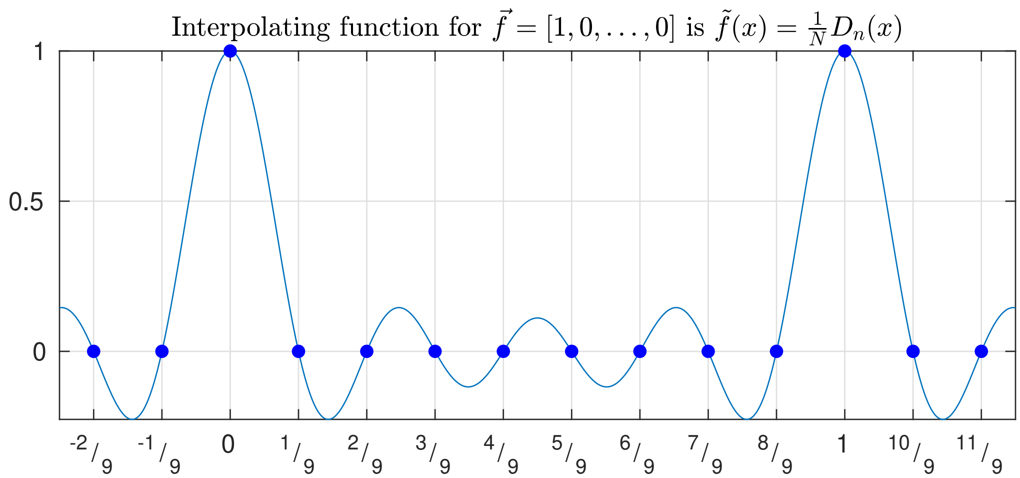
$$F(x)=\sum_{j=-n}^{n}\hat{F}_k u^{(k)}(x)$$

Assume that $f_0=1$ and $f_j=0$ if j is not a multiple of N. The discrete Fourier transform gives

$$\mathscr{F}_N[1,0,\dots,0]=N^{-1}[1,\dots,,1]$$

Hence we get with the Dirichlet kernel $D_n(x)$ defined in (37)

$$\tilde{f}(x)=N^{-1}\sum_{k=-n}^{n}u^{(k)}(x)=(2n+1)^{-1}D_n(x)=\begin{cases}1 & \text{for } x\in\mathbb{Z}\\ \frac{\sin((2n+1)\pi x)}{(2n+1)\sin(\pi x)} & \text{otherwise}\end{cases}$$



Here we see the graph of the function $\frac{1}{N}D_n(x)$ for $N=2n+1$ with $n=4$ (also see the graphs of $D_n(x)$ for $n=8$ on pages 19, 27).

Note that the function $N^{-1}D_n \in \mathscr{T}_n$ satisfies $N^{-1}D_n(\frac{j}{N}) = \begin{cases} 1 & \text{if } j \text{ is multiple of } N \\ 0 & \text{otherwise} \end{cases}$

Now consider $\vec{f} = [0,1,0,\ldots,0]$ where the values are shifted on notch to the right. Now the shifted function $N^{-1}D_n(x-\frac{1}{N})$ is the interpolating function.

Consider $\vec{f} = [3,-2,0,\ldots,0]$. Then $N^{-1}\left[3D_n(x) - 2D_n(x-\frac{1}{N})\right]$ is the interpolating function.

Hence we obtain an explicit formula for the interpolating function $\tilde{f}$:

$$\boxed{\tilde{f}(x) = N^{-1}\sum_{j=0}^{N-1} f_j \cdot D_n(x-\tfrac{j}{N})}$$

For even $N=2n$ the Fourier vector $[1,\ldots,,1]$ corresponds to the function

$$\begin{aligned} &\tfrac{1}{2}u^{(-n)} + u^{(-n+1)} + \cdots + u^{(n-1)} + \tfrac{1}{2}u^{(n)} = \tfrac{1}{2}q^{-n} + q^{-n+1} + \cdots + q^{n-1} + \tfrac{1}{2}q^n \\ &= \tfrac{1}{2}(q+1)\frac{q^n - q^{-n}}{q-1} = \tfrac{1}{2}(q^{\frac{1}{2}} + q^{-\frac{1}{2}})\frac{q^n - q^{-n}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = \boxed{\cos(\pi x)\cdot\frac{\sin(2n\pi x)}{\sin(\pi x)} =: D_{n,\cos}(x)} \end{aligned} \tag{59}$$

where $q := e^{2\pi i x}$. Therefore we obtain $\boxed{\tilde{f}(x) = N^{-1}\sum_{j=0}^{N-1} f_j \cdot D_{n,\cos}(x-\tfrac{j}{N})}$ for the interpolating function.

4.12 Efficient computation of $\mathscr{F}_N$ and $\mathscr{F}_N^{-1}$: Fast Fourier Transform

We will omit vector arrows in this section and write f, $\hat{f}$ for the vectors $\vec{f}, \vec{\hat{f}} \in \mathbb{C}^N$.

Assume we are given the vector $\hat{f} = \left[\hat{f}_0,\ldots,\hat{f}_{N-1}\right]^\top \in \mathbb{C}^N$ in the frequency domain. We want to find the vector $f = \mathscr{F}_N^{-1}f = \left[f_0,\ldots,f_{N-1}\right]^\top \in \mathbb{C}^N$ in the time domain. If we just use the formula

$$f_j = \sum_{k=0}^{N-1} \hat{f}_k e^{2\pi i k j/N} \tag{60}$$

we have to compute N multiplications and $N-1$ additions for each component, in addition to evaluating $e^{2\pi i k j/N}$. This means that **the computation of $\vec{f}$ costs at least N^2 multiplications** (plus a similar number of additions etc.). For a vector of length $N=1000$ this means that we have at least 10^6 operations. This means that it is not practical to compute discrete Fourier transforms of long vectors.

Fortunately there is a way to reorganize the computation in such a way that only $N\cdot\log_2 N$ operations are required. E.g., for $N=1024$ we have

$$N^2 \approx 10^6, \qquad N\cdot\log_2 N = 1024\cdot 10 \approx 10^4$$

Note that $e^{2\pi i k j/N} = \omega_N^{jk}$ where we define

$$\boxed{\omega_N := e^{2\pi i/N}}$$

Assume that N is even. Then we can **split up the sum in even and odd terms**: Using $\omega_{\frac{N}{2}} = \omega_N^2$ we obtain

$$f_j = \sum_{k=0}^{N-1}\omega_N^{jk}\hat{f}_k = \sum_{k=0}^{\frac{N}{2}-1}\omega_N^{j(2k)}\hat{f}_{2k} + \sum_{k=0}^{\frac{N}{2}-1}\omega_N^{j(2k+1)}\hat{f}_{2k+1} = \underbrace{\sum_{k=0}^{\frac{N}{2}-1}\omega_{\frac{N}{2}}^{jk}\hat{f}_{2k}}_{a_j} + \omega_N^j\underbrace{\sum_{k=0}^{\frac{N}{2}-1}\omega_{\frac{N}{2}}^{jk}\hat{f}_{2k+1}}_{b_j} \tag{61}$$

Recall that (60) implies that $f_{j+N} = f_j$. In the same way we get $a_{j+\frac{N}{2}} = a_j$ and $b_{j+\frac{N}{2}} = b_j$.

Note that

$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{\frac{N}{2}-1} \end{bmatrix} = \mathscr{F}_{\frac{N}{2}}^{-1} \begin{bmatrix} \hat{f}_0 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_{N-2} \end{bmatrix}, \qquad \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{\frac{N}{2}-1} \end{bmatrix} = \mathscr{F}_{\frac{N}{2}}^{-1} \begin{bmatrix} \hat{f}_1 \\ \hat{f}_3 \\ \vdots \\ \hat{f}_{N-1} \end{bmatrix}$$

After we computed the vectors $\begin{bmatrix} a_0 \\ \vdots \\ a_{\frac{N}{2}-1} \end{bmatrix}, \begin{bmatrix} b_0 \\ \vdots \\ b_{\frac{N}{2}-1} \end{bmatrix} \in \mathbb{C}^{\frac{N}{2}}$ we can use (61) to find $\begin{bmatrix} f_0 \\ \vdots \\ f_{N-1} \end{bmatrix} \in \mathbb{C}^N$: Using $a_{j+\frac{N}{2}} = a_j$, $b_{j+\frac{N}{2}} = b_j$ and $\omega_N^{\frac{N}{2}} = -1$ we obtain

$$\begin{bmatrix} f_0 \\ \vdots \\ f_{\frac{N}{2}-1} \\ \hline f_{\frac{N}{2}} \\ \vdots \\ f_{N-1} \end{bmatrix} = \begin{bmatrix} a_0 + \omega_N^0 b_0 \\ \vdots \\ a_{\frac{N}{2}-1} + \omega_N^{\frac{N}{2}-1} b_{\frac{N}{2}-1} \\ \hline a_{\frac{N}{2}} + \omega_N^{\frac{N}{2}} b_{\frac{N}{2}} \\ \vdots \\ a_{N-1} + \omega_N^{N-1} b_{N-1} \end{bmatrix} = \begin{bmatrix} a_0 + \omega_N^0 b_0 \\ \vdots \\ a_{\frac{N}{2}-1} + \omega_N^{\frac{N}{2}-1} b_{\frac{N}{2}-1} \\ \hline a_0 - \omega_N^0 b_0 \\ \vdots \\ a_{\frac{N}{2}-1} - \omega_N^{\frac{N}{2}-1} b_{\frac{N}{2}-1} \end{bmatrix} = \begin{bmatrix} a+c \\ a-c \end{bmatrix} \quad \text{where } a = \begin{bmatrix} a_0 \\ \vdots \\ a_{\frac{N}{2}-1} \end{bmatrix}, c := \begin{bmatrix} \omega_N^0 b_0 \\ \vdots \\ \omega_N^{\frac{N}{2}-1} b_{\frac{N}{2}-1} \end{bmatrix}$$

This gives the following **algorithm for computing $f = \mathscr{F}_N \hat{f}$:**

- let $a := \mathscr{F}_{\frac{N}{2}}^{-1} \begin{bmatrix} \hat{f}_0 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_{N-2} \end{bmatrix}, \quad b = \mathscr{F}_{\frac{N}{2}}^{-1} \begin{bmatrix} \hat{f}_1 \\ \hat{f}_3 \\ \vdots \\ \hat{f}_{N-1} \end{bmatrix}, \quad c_j := \omega_N^j b_j \quad \text{for } j = 0, \ldots, \frac{N}{2} - 1$
- let $f := \begin{bmatrix} a+c \\ a-c \end{bmatrix}$

With this algorithm we can compute $\mathscr{F}_N$ if N is a power of 2: We use recursion and compute $\mathscr{F}_{\frac{N}{2}}$, $\mathscr{F}_{\frac{N}{4}}, \ldots$ using this algorithm and note that $\mathscr{F}_1 \hat{f} = f$.

Here is a Matlab implementation of this algorithm: We assume that `fh` is a row vector where the length is a power of 2:

```
function f = ifour_rec(fh)
N = length(fh);
if N>1
  if mod(N,2)==1
    error('length must be power of 2')
  end
  a = ifour_rec(fh(1:2:end));
  c = ifour_rec(fh(2:2:end)).*exp(2i*pi/N*(0:N/2-1));
  f = [a+c,a-c];
else                                                  % case N=1
  f = fh;
end
```

Let C_N denote the number of operations needed to compute the vector $\mathscr{F}_N^{-1} \hat{f}$. We need to compute 2 transforms of length $\frac{N}{2}$ to compute the vectors a, b. Then we need about N additional operations to compute $f_0, \ldots, f_{N-1}$ from a, b. Therefore we have for even N that

$$C_N = 2C_{N/2} + N$$

We obtain the following costs for $N = 1, 2, 4, 8, \ldots$

$$\begin{aligned} C_1 &= 0 \\ C_2 &= 0 \cdot 2 + 2 = 1 \cdot 2 \\ C_4 &= 1 \cdot 2 \cdot 2 + 4 = 2 \cdot 4 \\ C_8 &= 2 \cdot 4 \cdot 2 + 8 = 3 \cdot 8 \\ C_{16} &= 3 \cdot 8 \cdot 2 + 16 = 4 \cdot 16 \\ C_{32} &= 4 \cdot 16 \cdot 2 + 32 = 5 \cdot 32 \end{aligned}$$

yielding a cost $\boxed{C_N = N \cdot \log_2 N}$ if N is a power of 2.

The same idea works if N **is a multiple of 3:** Then we split the sum (60) into three parts: $k = 0, 3, 6, 9, \ldots$ and $k = 1, 4, 7, \ldots$ and $k = 2, 5, 8, \ldots$

$$\begin{aligned} f_j = \sum_{k=0}^{N-1} \omega_N^{jk} \hat{f}_k &= \sum_{k=0}^{N/3-1} \omega_N^{j(3k)} \hat{f}_{3k} + \sum_{k=0}^{N/3-1} \omega_N^{j(3k+1)} \hat{f}_{3k+1} + \sum_{k=0}^{N/3-1} \omega_N^{j(3k+2)} \hat{f}_{3k+2} \\ &= \sum_{k=0}^{N/3-1} \omega_{N/3}^{jk} \hat{f}_{3k} + \omega_N^{j} \sum_{k=0}^{N/3-1} \omega_{N/3}^{jk} \hat{f}_{3k+1} + \omega_N^{2j} \sum_{k=0}^{N/3-1} \omega_{N/3}^{jk} \hat{f}_{3k+1} \end{aligned} \tag{62}$$

Let $D := \begin{bmatrix} 1 & & & \\ & \omega_N & & \\ & & \ddots & \\ & & & \omega_N^{N/3-1} \end{bmatrix}$ and define

$$a^{(0)} := \mathscr{F}_{N/3}^{-1} \begin{bmatrix} \hat{f}_0 \\ \hat{f}_3 \\ \vdots \\ \hat{f}_{N-3} \end{bmatrix}, \quad a^{(1)} := D \mathscr{F}_{N/3}^{-1} \begin{bmatrix} \hat{f}_1 \\ \hat{f}_4 \\ \vdots \\ \hat{f}_{N-2} \end{bmatrix}, \quad a^{(2)} := D^2 \mathscr{F}_{N/3}^{-1} \begin{bmatrix} \hat{f}_2 \\ \hat{f}_5 \\ \vdots \\ \hat{f}_{N-1} \end{bmatrix}$$

Then (62) gives using $\omega_N^{N/3} = \omega_3$ and $\omega_3^4 = \omega_3$

$$\begin{bmatrix} f_0 \\ \vdots \\ f_{N-1} \end{bmatrix} = \begin{bmatrix} a^{(0)} + a^{(1)} + a^{(2)} \\ a^{(0)} + \omega_3 a^{(1)} + \omega_3^2 a^{(2)} \\ a^{(0)} + \omega_3^2 a^{(1)} + \omega_3 a^{(2)} \end{bmatrix}$$

If N is a multiple of 3 the cost of $\mathscr{F}_N$ is

$$C_N = 3 C_{N/3} + 2N$$

Matlab's `fft` and `ifft` are implemented using the **FFTW library** ("Fastest Fourier Transform in the West") which uses a number of similar techniques to allow arbitrary values of N.

- this is most efficient if N is a power of 2, or a product of small primes like $2, 3, 5, 7$
- even if N is a large prime the cost is $O(N \log N)$