Review of Taylor's theorem

1 Introduction

NOTATION:

- assignment: $\boxed{x := y}$ indicates that x is defined to be y. E.g., we write $x := x+1$ to indicate that we increase the value of x by 1.
- $\boxed{x = y}$ means that the values x, y (which we defined earlier) are equal
- **in most programming languages** (including **Matlab**):
  assignment $x := y$ is written as **`x=y`**, this means that the variable x gets a new value.
  expression $x = y$ is written as **`x==y`**, this expression evaluates to **`true`** or **`false`** (this can be used in an **`if`** statement)
  Matlab command line:

```
>> x=5;      % define x to be 5
>> x==x+1    % returns false, in Matlab shown as 0 with type logical
ans =
  logical
   0
>> x=x+1     % increase x by 1
x =
     6
```

- $\boxed{x \overset{!}{=} y}$ means that we impose a **new condition which we want to hold**. We will then find parameters for which this condition holds.
  Example: find the extreme points of the function $f(x) = x^3 - 3x$
  We first find the derivative $f'(x) = 3x^2 - 3$. At an extreme point we want to have

$$f'(x) \overset{!}{=} 0 \qquad \text{(necessary condition for extreme point)}$$
$$3x^2 - 3 = 0 \quad \Longleftrightarrow \quad x^2 = 1 \quad \Longleftrightarrow \quad x = 1 \text{ OR } x = -1$$

For numerical computations we want to approximate a function value $y = f(x)$ by a value $\tilde{y}$ which we can compute in finitely many operations on our machine.

A key tool for approximation is the **Taylor polynomial**. Most of the approximation techniques in this course are based on Taylor approximation.

Therefore we need to review the key ideas of Taylor approximation:

- **approximate** $y = f(x)$**:**
  pick x_0, polynomial degree n
  find the **Taylor polynomial** $\boxed{p_n(x)}$
- **approximation error** $y - \tilde{y}$**:**
  we have $y = \tilde{y} + R_{n+1}(x)$ and a formula for the **remainder term** $\boxed{R_{n+1}(x)}$

This will be on the first assignment and on the first exam.

2 Taylor polynomial $p_n(x)$

We want to approximate the function value $f(x)$. We only know the values $f(x_0)$, $f'(x_0)$, $f''(x_0), \dots$ at a nearby point x_0.

An approximation for $f(x)$ is given by the **Taylor polynomial**

$$\boxed{p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + f^{(n)}(x_0)\frac{(x - x_0)^n}{n!}} \tag{1}$$

Note: We need to choose x_0 such that

- x_0 is close to x
- we know the values $f(x_0)$, $f'(x_0)$, $f''(x_0)$, …

Example 1

Approximate $y = \frac{1}{2.02}$ using a Taylor polynomial $p_3(x)$.

1. **Figure out what f, x, x_0, n are:**
   Here the function is $f(x) = x^{-1}$, we want to evaluate this at $x = 2.02$.
   We need to pick x_0 close to x where we can easily find $f(x_0), \dots, f^{(n)}(x_0)$: we use $x_0 = 2$
   Here we are told to use $n = 3$. Often we need to figure out n so that the approximation error is sufficiently small, see Example 2 below.
2. **Find the derivatives $f', \dots, f^{(n)}$:** $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -6x^{-4}$
3. **Evaluate $f, f', \dots, f^{(n)}$ at x_0:** $f(x_0) = \frac{1}{2}$, $f'(x_0) = -\frac{1}{4}$, $f''(x_0) = \frac{2}{8} = \frac{1}{4}$, $f'''(x_0) = -\frac{6}{16} = -\frac{3}{8}$
4. **Write down Taylor polynomial $p_n(x)$** as an expression of x:

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + f^{(n)}(x_0)\frac{(x-x_0)^n}{n!} = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3$$

5. **Plug in the given value of x:** here $x = 2.02$ and we obtain

$$\tilde{y} = p_n(x_0) = \frac{1}{2} - \frac{1}{4}0.02 + \frac{1}{8}0.02^2 - \frac{1}{16}0.02^3 = \underbrace{0.5 - 0.005}_{0.495} + \underbrace{0.00005 - 0.0000005}_{0.0000495} = 0.4950495$$

Important: do these five steps separately. Trying to do everything at once usually leads to mistakes.

Homework problem 1(a): Find $p_5(x)$ as a function of x by hand. Then plug in $x = 2.02$ and simplify. You can use Matlab to evaluate the resulting expression.

3 Remainder term R_{n+1}

Taylors theorem gives a formula **remainder term** $R_{n+1} = f(x) - p_n(x)$.

Theorem 1. *Assume that f has continuous derivatives up to order $n+1$ between x and x_0. Then*

$$f(x) = p_n(x) + R_{n+1}$$

with the Taylor polynomial(1) *and the* ***remainder term***

$$\boxed{R_{n+1} = f^{(n+1)}(t)\frac{(x-x_0)^{n+1}}{(n+1)!}} \qquad (2)$$

where $\boxed{t \text{ is between } x \text{ and } x_0}$

This shows that

$$|f(x) - p_n(x)| \le C\,|x - x_0|^{n+1}$$

I.e., the error decreases quickly as $x$gets closer to x_0.

Example 1 (continued)

Use the remainder term to find an upper bound for the error $|y - \tilde{y}| = |f(x) - p_3(x)| \le \cdots$

Step 6: Find the remainder term R_{n+1} containing the unknown value t The fourth derivative is $f^{(4)}(x) = 24x^{-5}$. We have

$$f(x) - p_3(x) = R_4 = f^{(4)}(t)\frac{(x-x_0)^4}{4!} = 24t^{-5} \times \frac{0.02^4}{24} = t^{-5} \times 0.02^4$$

Step 7: Use that $t$is between x and x_0 to find an upper bound $|R_{n+1}| \leq \cdots$ Here t is between $x = 2.02$ and $x_0 = 2$. **Since the function t^{-5} is decreasing on the interval** $[2, 2.02]$ and **we get the largest value at the left endpoint 2**:

$$|t^{-5}| \leq 2^{-5}$$
$$|y - \tilde{y}| \leq \left|t^{-5} \times 0.02^4\right| \leq 2^{-5} \times 0.02^4 = \tfrac{1}{2} \times 0.01^4 = \tfrac{1}{2} \times 10^{-8} = 5 \times 10^{-9}$$

Note that the actual error is

$$|y - \tilde{y}| = \left|\frac{1}{2.02} - 0.4950495\right| \approx 4.95 \cdot 10^{-9}$$

This is indeed $\leq$ our upper bound, but very close to it.

Homework problem 1(b): Find an upper bound $|f(x) - p_5(x)| \leq \cdots$ by hand.

Find the actual error $|f(x) - p_5(x)|$ using Matlab. Compare this with the upper bound.

Example 2

We want to compute $y = \sqrt{9.1}$. We have a simple calculator which can only add, subtract, multiply and divide. How can we use the calculator to compute $\tilde{y}$ with $|\tilde{y} - y| \leq 10^{-6}$?

Solution

Here $f(x) = \sqrt{x}$ and $x = 9.1$. We first try $n = 2$ and check if our error bound is small enough.

Step 1: Choose x_0. Here we should use $x_0 = 9$ since it is close to $x = 9.1$ and we can evaluate $f(x_0), f'(x_0), f''(x_0), \ldots$

Step 2: Find the derivatives f', f'', f''', ... Here $f(x) = x^{1/2}$, so $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$, $f'''(x) = \frac{3}{8}x^{-5/2}$, $f^{(4)}(x) = -\frac{15}{16}x^{-7/2}$ etc.

Step 3: Evaluate $f(x_0)$, $f'(x_0)$, $f''(x_0)$, $f'''(x_0)$, ... Here $x_0 = 9$, so we obtain

$$f(x_0) = 9^{1/2} = 3, \quad f'(x_0) = \tfrac{1}{2} \cdot 9^{-1/2} = \tfrac{1}{2} \cdot 3^{-1} = \tfrac{1}{6}, \quad f''(x_0) = -\tfrac{1}{4} \cdot 9^{-3/2} = -\tfrac{1}{4} \cdot 3^{-3} = -\tfrac{1}{108}$$

Step 4: Write down the Taylor polynomial $p_2(x)$ in terms of x:

$$p_2(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\tfrac{1}{2}(x - x_0)^2 = 3 + \frac{1}{6} \cdot (x - 9) + \frac{-1}{108} \cdot \frac{1}{2} \cdot (x - 9)^2$$

Step 5: Plug in $x = 9.1$:

$$\tilde{y} = p_2(9.1) = 3 + \frac{1}{6} \cdot 0.1 + \frac{-1}{108} \cdot \frac{1}{2} \cdot 0.1^2 = 3 + \frac{1}{60} + \frac{-1}{21600} \approx 3.01662037037$$

Step 6: Find the remainder term R_{n+1} containing the unknown value t: Here $n = 2$ and we have

$$R_3 = f'''(t) \cdot \frac{1}{(n+1)!} \cdot (x - x_0)^{n+1} = \frac{3}{8} \cdot t^{-5/2} \cdot \frac{1}{3!} \cdot (9.1 - 9)^3 = \frac{3}{8} \cdot \frac{1}{6} \cdot \frac{1}{10^3} \cdot t^{-5/2}$$

Step 7: Use that t is between x and x_0 to find an upper bound $|R_{n+1}| \leq \cdots$
Here t is between $x = 9.1$ and $x_0 = 9$. Since $t^{-5/2}$ is a decreasing function we get $\left|t^{-5/2}\right| \leq 9^{-5/2} = (9^{1/2})^{-5}$ and

$$|R_3| \leq \frac{3}{8} \cdot \frac{1}{6} \cdot \frac{1}{10^3} \cdot 3^{-5} \approx 2.572 \cdot 10^{-7}$$

Therefore we have $|\tilde{y} - y| \leq 2.573 \cdot 10^{-7} \leq 10^{-6}$. Therefore $n = 2$ is sufficient.
If we want to achieve $|\tilde{y} - y| \leq 10^{-7}$ we would try $n = 3$.