Practice problems: Solutions

1. Assume that we do the following operations in Matlab. Give an upper bound for the relative error of the computed result.

(a) `y = 1000.2 - 1000.1`
$\hat{y}_1 := fl(1000.2)$, $\hat{y}_2 := fl(1000.1)$, $\tilde{y} := \hat{y}_1 - \hat{y}_2$, $\hat{y} := fl(\tilde{y})$.
Let $\epsilon_{\hat{y}_1} = \frac{\hat{y}_1 - y_1}{y_1}$ etc. Then $|\epsilon_{\hat{y}_1}| \le \epsilon_M$, $|\epsilon_{\hat{y}_2}| \le \epsilon_M$, $|\epsilon_{\tilde{y}}| \le \frac{|y_1|}{|y_1 - y_2|} |\epsilon_{\hat{y}_1}| + \frac{|y_2|}{|y_1 - y_2|} |\epsilon_{\hat{y}_2}| \le 20003\epsilon_M$,
$|\epsilon_{\hat{y}}| \le |\epsilon_{\tilde{y}}| + \epsilon_M \le 20004\epsilon_M \approx 2 \cdot 10^{-12}$

(b) `y = exp(.001) - 1`
$\hat{y}_1 := fl(.001)$, $\tilde{y}_2 := \exp(\hat{y}_1)$, $\hat{y}_2 := fl(\tilde{y}_2)$, $\tilde{y} := \hat{y}_2 - 1$, $\hat{y} := fl(\tilde{y})$.
$|\epsilon_{\hat{y}_1}| \le \epsilon_M$, $|\epsilon_{\tilde{y}_2}| \le \left| \frac{y_1 \exp(y_1)}{\exp(y_1)} \right| |\epsilon_{\hat{y}_1}| \le .001\epsilon_M$, $|\epsilon_{\hat{y}_2}| \le |\epsilon_{\tilde{y}_2}| + \epsilon_M \le 1.001\epsilon_M$, $|\epsilon_{\tilde{y}}| \le \left| \frac{y_2}{y_2 - 1} \right| |\epsilon_{\hat{y}_2}| \le 1001.5\epsilon_M$,
$|\epsilon_{\hat{y}}| \le |\epsilon_{\tilde{y}}| + \epsilon_M \le 1002.5\epsilon_M \approx 10^{-13}$.

(c) How can we get a more accurate result for (b)?
Use the Taylor series for $f(x) = e^x - 1$ about $x_0 = 0$: E.g., $f(x) \approx \tilde{y} = 0 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$ and $\left| \frac{\tilde{y} - y}{y} \right| \approx \frac{t^6/6!}{x} \le \frac{10^{-18}/720}{10^{-3}} \approx 1.4 \cdot 10^{-18}$.

2. Consider the matrix $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 4 \\ 4 & 1 & 2 \end{pmatrix}$,

(a) Apply Gaussian elimination using the pivot candidate with the largest absolute value to find the matrices L, U and the vector p .
$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 1/2 & 2/7 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 7/4 & 7/2 \\ 0 & 0 & 2 \end{pmatrix}$, $p = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

(b) Rearrange the rows of the matrix L to obtain L_1 with $L_1 U = A$.
$L_1 = \begin{pmatrix} 1/4 & 1 & 0 \\ 1/2 & 2/7 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

(c) Using L, U, p solve the linear system $Ax = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

Solve $Ly = b'$: $y = \begin{pmatrix} -1 \\ 9/4 \\ 6/7 \end{pmatrix}$, solve $Ux = y$: $x = \begin{pmatrix} -4/7 \\ 3/7 \\ 3/7 \end{pmatrix}$

(d) We want to solve $Ax = b$. But we only have an approximate vector $\tilde{b}$ with $\left| \frac{\tilde{b}_j - b_j}{b_j} \right| \le .01$ and solve the linear system $A\tilde{x} = \tilde{b}$. What can you say about $\frac{\|\tilde{x} - x\|_\infty}{\|x\|_\infty}$ if $\|A^{-1}\|_\infty \le 2.5$?
By taking the maximum of $\left|\tilde{b}_j - b_j\right| \le .01\, |b_j|$ we get $\left\|\tilde{b}_j - b\right\|_\infty \le .01\, \|b\|_\infty$.

$$\frac{\|\tilde{x} - x\|_\infty}{\|x\|_\infty} \le \|A\|_\infty \left\|A^{-1}\right\|_\infty \frac{\left\|\tilde{b} - b\right\|_\infty}{\|b\|_\infty} \le 7 \cdot 2.5 \cdot 0.01 = .175$$