Practice problems and solutions for Exam 2

1. We are given the following information about $f(x)$:

$$f(0) = 2, \quad f(1) = 1, \quad f(3) = 0, \quad f(4) = 1$$

(a) Write down the divided difference table. Find interpolating polynomial in Newton form (i) for the nodes in the order $0, 1, 3, 4$, (ii) for the nodes in the order $4, 3, 1, 0$.

Divided difference table: $x_1 = 0$, $x_2 = 1$, $x_3 = 3$, $x_4 = 4$

$$\begin{array}{llll} f[x_1] = 2 & f[x_1,x_2] = -1 & f[x_1,x_2,x_3] = \frac{1}{6} & f[x_1,x_2,x_3,x_4] = \frac{1}{12} \\ f[x_2] = 1 & f[x_2,x_3] = -\frac{1}{2} & f[x_2,x_3,x_4] = \frac{1}{2} & \\ f[x_3] = 0 & f[x_3,x_4] = 1 & & \\ f[x_4] = 1 & & & \end{array}$$

$$\begin{aligned} p(x) &= f[x_1] + f[x_1,x_2](x-x_1) + f[x_1,x_2,x_3](x-x_1)(x-x_2) + f[x_1,x_2,x_3,x_4](x-x_1)(x-x_2)(x-x_3) \\ &= 2 + (-1)\cdot(x-0) + \tfrac{1}{6}(x-0)(x-1) + \tfrac{1}{12}\cdot(x-0)(x-1)(x-3) \\ p(x) &= f[x_4] + f[x_3,x_4](x-x_4) + f[x_2,x_3,x_4](x-x_3)(x-x_4) + f[x_1,x_2,x_3,x_4](x-x_2)(x-x_3)(x-x_4) \\ &= 1 + 1\cdot(x-4) + \tfrac{1}{2}(x-4)(x-3) + \tfrac{1}{12}\cdot(x-4)(x-3)(x-1) \end{aligned}$$

(b) Assume we know that the 4th derivative satisfies $\left|f^{(4)}(x)\right| \le 10$ for $x \in [0,4]$. Find an upper bound for $|f(2) - p(2)|$.

Let $\tilde{x} = 2$. The error formula states that there exists $t \in (0,4)$ such that

$$\begin{aligned} |f(\tilde{x}) - p(\tilde{x})| &= \frac{\left|f^{(4)}(t)\right|}{4!} |(\tilde{x}-x_1)(\tilde{x}-x_2)(\tilde{x}-x_3)(\tilde{x}-x_4)| \\ &\le \frac{10}{24}\cdot 2\cdot 1\cdot 1\cdot 2 = \frac{5}{3} \end{aligned}$$

2. Consider the (x,y) data points $(-1,2)$, $(1,1)$, $(2,0)$. We want to fit the data with a function $g(x) = c_1 + c_2 x^2$

(a) Find the best least squares fit by hand.

Here $y = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. With $\phi_1(x) = 1$ and $\phi_2(x) = x^2$ we have $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 4 \end{bmatrix}$. The normal equations are $A^\top A c = A^\top y$:

$$\begin{bmatrix} 3 & 6 \\ 6 & 18 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

which gives $c_1 = 2$, $c_2 = -\frac{1}{2}$.

(b) Write a Matlab program which uses `qr` to solve this problem.

```
x = [-1;1;2]; y=[2;2;0];
A = [x.^0,x.^2];
[Q,R] = qr(A,0);
d = Q'*y;
c = R\d;
```

3. We want to find $c \in \mathbb{R}^2$ such that $\|Ac - y\|_2$ is minimal. Here $y = [1,2,3,0]^\top$, and for the matrix A we have the QR decomposition $A = QR$ with

$$Q = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}.$$

Use this to find the solution vector c. DO **NOT** COMPUTE $A = QR$. DO **NOT** USE THE NORMAL EQUATIONS FOR THE MATRIX A.

We want to find $c \in \mathbb{R}^2$ such that $\|QRc - y\|_2$ is minimal. Let $d := Rc$, then we have to find $d \in \mathbb{R}^2$ such that $\|Qd - y\|_2$ is minimal. The normal equations give that $Q^\top Q d = Q^\top y$. The matrix Q has orthonormal columns, hence $Q^\top Q = I$ is the identity matrix. So we get

$$d = Q^\top y = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

and we obtain c by solving the upper triangular system $Rc = d$

$$\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \Longrightarrow \quad c = \begin{bmatrix} 2.5 \\ 2 \end{bmatrix}$$

4. We want to find x such that $x + x^5 = 3$.

(a) Perform one step of the bisection method with $a_0 = 1$, $b_0 = 2$. Find k such that $|b_k - a_k| \le 10^{-6}$.
$f(x) = x^5 + x - 3$, $f(a_0) = -1 < 0$, $f(b_0) = 31 > 0$, $c_0 = (a_0 + b_0)/2 = 1.5$, $f(1.5) = 1.5^5 + 1.5 - 3 = 1.5^5 - 1.5 > 0$, hence $[a_1, b_1] = [a_0, c_0] = [1, 1.5]$. So we have $x_* \in (1, 1.5)$.
We have $|b_k - a_k| = 2^{-k}|b_0 - a_0| = 2^{-k}$. We have

$$2^{-k} \le 10^{-6} \iff (-k)\log 2 \le \log(10^{-6}) \iff k \ge \frac{\log(10^6)}{\log 2} = 19.93,$$

hence we need $k \ge 20$.

(b) Perform one step of the secant method with $x_0 = 1$, $x_1 = 2$ to find x_2.
$x_2 = x_1 - f(x_1)\dfrac{x_1 - x_0}{f(x_1) - f(x_0)} = 2 - 31\dfrac{2-1}{31-(-1)} = 2 - \frac{31}{32} = 1 + \frac{1}{32}$

(c) Will the Newton method converge if we start with x_0 sufficiently close to the solution x_*? Explain.
We showed: If $f, f', f''$are continuous and $f'(x_*) \ne 0$, then the Newton method converges for x_0 sufficiently close to x_*. Here $f(x) = x^5 + x - 3$, $f'(x) = 5x^4 + 1$. We know from the intermediate value theorem that there is a root in the interval $(1, 2)$. Since $f'(x) \ge 1 > 0$ there is a unique root, and we must have $f'(x_*) > 0$. So all the assumptions of the theorem are satisfied.

5. Consider the nonlinear system

$$x_1 + x_1x_2 + x_2 = 2, \qquad x_1 - x_2 - x_1x_2^2 = 0$$

(a) Perform one step of the Newton method starting with initial guess $x^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

We have $\mathbf{f}(\mathbf{x}) = \begin{bmatrix} x_1 + x_1x_2 + x_2 - 2 \\ x_1 - x_2 - x_1x_2^2 \end{bmatrix}$ with the Jacobian matrix $f'(\mathbf{x}) = \begin{bmatrix} 1+x_2 & x_1+1 \\ 1-x_2^2 & -1-2x_1x_2 \end{bmatrix}$. For $\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we get $\mathbf{y} = \mathbf{f}(\mathbf{x}^{(0)}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $A = f'(\mathbf{x}^{(0)}) = \begin{bmatrix} 2 & 2 \\ 0 & -3 \end{bmatrix}$. Solving the linear system $A\mathbf{d} = -\mathbf{y}$ gives $\mathbf{d} = \begin{bmatrix} -1/6 \\ -1/3 \end{bmatrix}$ and the new approximation $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \mathbf{d} = \begin{bmatrix} 5/6 \\ 2/3 \end{bmatrix}$.

(b) Write a Matlab program which uses the Newton method to find a solution, starting with initial guess $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The program should print out the approximation for $\mathbf{x}$ after each iteration.

```
f = @(x) [ x(1)+x(1)*x(2)+x(2)-2 ; x(1)-x(2)-x(1)*x(2)^2 ];
fp = @(x) [ 1+x(2), x(1)+1; 1-x(2)^2, -1-2*x(1)*x(2)];
x = [1;1];
while 1
  b = f(x); A = fp(x);
  d = -A\b;
  x = x + d                    % prints out x for each iteration
  if norm(d)<1e-14
    break
  end
end
```

6. Let $g(x) = \frac{1}{4}\begin{bmatrix} 1 + x_2 + \cos(x_1 + x_2) \\ 1 + x_1 + \sin(x_1 - x_2) \end{bmatrix}$

(a) Let $D = [0, 3] \times [0, 3]$. We need to check the three assumptions of the contraction mapping theorem:
(1.) D is closed - true since the boundary of the square is included in D.
(2.) Show: $x \in D \implies g(x) \in D$. Let $y = g(x)$. Then for $x_1, x_2 \in [0, 1]$ we have

$$0 \le 2 + 0 - 1 \le 4y_1 = 1 + x_2 + \cos(x_1 + x_2) \le 1 + 1 + 1 \le 12$$
$$0 \le 1 + 0 - 1 \le 4y_2 = 1 + x_1 + \sin(x_1 - x_2) \le 1 + 1 + 1 \le 12$$

(3.) Show: g is contraction on D. Note that D is convex. The Jacobian is

$$g'(x) = \frac{1}{4}\begin{bmatrix} -\sin(x_1+x_2) & 1-\sin(x_1+x_2) \\ 1+\cos(x_1-x_2) & -\cos(x_1-x_2) \end{bmatrix}$$

and we have

$$\|g'(x)\|_\infty \leq \frac{1}{4}\max\{1+(1+1),(1+1)+1\} = \frac{3}{4} =: q < 1$$

Now we can apply the contraction mapping theorem and obtain that the nonlinear system has a unique solution x^* in the set D.

(b) For $x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ we obtain $x^{(1)} = g(x^{(0)}) = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$. The a-posteriori estimate gives

$$\left\|x^{(1)} - x^*\right\|_\infty \leq \frac{q}{1-q}\left\|x^{(1)} - x^{(0)}\right\|_\infty = \frac{3/4}{1/4}\cdot\frac{1}{2} = \frac{3}{2}$$

which means that $\left|x_1^* - \frac{1}{2}\right| \leq \frac{3}{2}$, $\left|x_2^* - \frac{1}{4}\right| \leq \frac{3}{2}$, i.e.,

$$x^* \in D_1 = [-1,2]\times[-\frac{5}{4},\frac{7}{4}]$$

Since we also know that $x^* \in D$ we actually know that

$$x^* \in D_1 \cap D = [0,2]\times[0,\frac{7}{4}].$$