Practice problems and solutions for Exam 2

1. We are given the following information about $f(x)$:

$$f(1) = 2, \quad f'(1) = 1, \quad f(3) = 2, \quad f'(3) = -1$$

(a) Write down the divided difference table and the interpolating polynomial in Newton form.
Divided difference table: $x_1 = x_2 = 1$, $x_3 = x_4 = 3$

$$\begin{array}{llll} f[x_1] = 2 & f[x_1,x_2] = 1 & f[x_1,x_2,x_3] = -\frac{1}{2} & f[x_1,x_2,x_3,x_4] = 0 \\ f[x_2] = 2 & f[x_2,x_3] = 0 & f[x_2,x_3,x_4] = -\frac{1}{2} & \\ f[x_3] = 2 & f[x_3,x_4] = -1 & & \\ f[x_4] = 2 & & & \end{array}$$

$$\begin{aligned} p(x) &= f[x_1] + f[x_1,x_2](x-x_1) + f[x_1,x_2,x_3](x-x_1)(x-x_2) + f[x_1,x_2,x_3,x_4](x-x_1)(x-x_2)(x-x_3) \\ &= 2 + 1\cdot(x-1) + (-\tfrac{1}{2})(x-1)^2 + 0\cdot(x-1)^2(x-3) \\ p(x) &= f[x_4] + f[x_3,x_4](x-x_4) + f[x_2,x_3,x_4](x-x_3)(x-x_4) + f[x_1,x_2,x_3,x_4](x-x_2)(x-x_3)(x-x_4) \\ &= 2 + (-1)(x-3) + (-\tfrac{1}{2})(x-3)^2 + 0\cdot(x-1)(x-3)^2 \end{aligned}$$

(b) Assume we know that the 4th derivative satisfies $\left|f^{(4)}(x)\right| \le 10$ for $x \in [1,2]$. Find an upper bound for $|f(1.5) - p(1.5)|$.
Let $\tilde{x} = 1.5$. The error formula states that there exists $t \in (1,2)$ such that

$$\begin{aligned} |f(\tilde{x}) - p(\tilde{x})| &= \frac{\left|f^{(4)}(t)\right|}{4!}\left|(\tilde{x}-x_1)(\tilde{x}-x_2)(\tilde{x}-x_3)(\tilde{x}-x_4)\right| \\ &\le \frac{10}{24}(1.5-1)^2(1.5-3)^2 = \frac{10}{24}\cdot\frac{1}{4}\cdot\frac{9}{4} \end{aligned}$$

2. We want to find x such that $x + x^5 = 3$.

(a) Perform one step of the bisection method with $a_0 = 1$, $b_0 = 2$. Find k such that $|b_k - a_k| \le 10^{-6}$.
$f(x) = x^5 + x - 3$, $f(a_0) = -1 < 0$, $f(b_0) = 31 > 0$, $c_0 = (a_0 + b_0)/2 = 1.5$, $f(1.5) = 1.5^5 + 1.5 - 3 = 1.5^5 - 1.5 > 0$, hence $[a_1, b_1] = [a_0, c_0] = [1, 1.5]$. So we have $x_* \in (1, 1.5)$.
We have $|b_k - a_k| = 2^{-k}|b_0 - a_0| = 2^{-k}$. We have

$$2^{-k} \le 10^{-6} \iff (-k)\log 2 \le \log(10^{-6}) \iff k \ge \frac{\log(10^6)}{\log 2} = 19.93,$$

hence we need $k \ge 20$.

(b) Perform one step of the secant method with $x_0 = 1$, $x_1 = 2$ to find x_2.
$x_2 = x_1 - f(x_1)\dfrac{x_1 - x_0}{f(x_1) - f(x_0)} = 2 - 31\dfrac{2-1}{31-(-1)} = 2 - \frac{31}{32} = 1 + \frac{1}{32}$

(c) Will the Newton method converge if we start with x_0 sufficiently close to the solution x_*? Explain.
We showed: If f, f', f'' are continuous and $f'(x_*) \neq 0$, then the Newton method converges for x_0 sufficiently close to x_*. Here $f(x) = x^5 + x - 3$, $f'(x) = 5x^4 + 1$. We know from the intermediate value theorem that there is a root in the interval $(1,2)$. Since $f'(x) \ge 1 > 0$ there is a unique root, and we must have $f'(x_*) > 0$. So all the assumptions of the theorem are satisfied.

3. Consider the nonlinear system

$$x_1 + x_1x_2 + x_2 = 2, \qquad x_1 - x_2 - x_1x_2^2 = 0$$

Perform one step of the Newton method starting with initial guess $x^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

We have $\mathbf{f}(\mathbf{x}) = \begin{bmatrix} x_1 + x_1x_2 + x_2 - 2 \\ x_1 - x_2 - x_1x_2^2 \end{bmatrix}$ with the Jacobian matrix $f'(\mathbf{x}) = \begin{bmatrix} 1 + x_2 & x_1 + 1 \\ 1 - x_2^2 & -1 - 2x_1x_2 \end{bmatrix}$. For $\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we get $\mathbf{y} = \mathbf{f}(\mathbf{x}^{(0)}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $A = f'(\mathbf{x}^{(0)}) = \begin{bmatrix} 2 & 2 \\ 0 & -3 \end{bmatrix}$. Solving the linear system $A\mathbf{d} = -\mathbf{y}$ gives $\mathbf{d} = \begin{bmatrix} -1/6 \\ -1/3 \end{bmatrix}$ and the new approximation $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \mathbf{d} = \begin{bmatrix} 5/6 \\ 2/3 \end{bmatrix}$.

4. Let $g(x) = \frac{1}{4}\begin{bmatrix} 1 + x_2 + \cos(x_1 + x_2) \\ 1 + x_1 + \sin(x_1 - x_2) \end{bmatrix}$

(a) Let $D = [0,3] \times [0,3]$. We need to check the three assumptions of the contraction mapping theorem:
(1.) D is closed - true since the boundary of the square is included in D.
(2.) Show: $x \in D \implies g(x) \in D$. Let $y = g(x)$. Then for $x_1, x_2 \in [0,1]$ we have

$$\begin{aligned} 0 \le 2 + 0 - 1 \le 4y_1 &= 1 + x_2 + \cos(x_1 + x_2) \le 1 + 1 + 1 \le 12 \\ 0 \le 1 + 0 - 1 \le 4y_2 &= 1 + x_1 + \sin(x_1 - x_2) \le 1 + 1 + 1 \le 12 \end{aligned}$$

(3.) Show: g is contraction on D. Note that D is convex. The Jacobian is

$$g'(x) = \frac{1}{4} \begin{bmatrix} -\sin(x_1 + x_2) & 1 - \sin(x_1 + x_2) \\ 1 + \cos(x_1 - x_2) & -\cos(x_1 - x_2) \end{bmatrix}$$

and we have

$$\|g'(x)\|_\infty \le \frac{1}{4} \max\{1 + (1+1), (1+1) + 1\} = \frac{3}{4} =: q < 1$$

Now we can apply the contraction mapping theorem and obtain that the nonlinear system has a unique solution x^* in the set D.

(b) For $x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ we obtain $x^{(1)} = g(x^{(0)}) = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$. The a-posteriori estimate gives

$$\left\| x^{(1)} - x^* \right\|_\infty \le \frac{q}{1-q} \left\| x^{(1)} - x^{(0)} \right\|_\infty = \frac{3/4}{1/4} \cdot \frac{1}{2} = \frac{3}{2}$$

which means that $\left|x_1^* - \frac{1}{2}\right| \le \frac{3}{2}$, $\left|x_2^* - \frac{1}{4}\right| \le \frac{3}{2}$, i.e.,

$$x^* \in D_1 = [-1, 2] \times [-\frac{5}{4}, \frac{7}{4}]$$

Since we also know that $x^* \in D$ we actually know that

$$x^* \in D_1 \cap D = [0, 2] \times [0, \frac{7}{4}].$$