Practice Problem: Solution

1. Consider the initial value problem

$$y'' + y' + y = t, \qquad y(1) = 1, \quad y'(1) = 2$$

(a) Perform one step of the *Euler method* with $h = 1$ and give the resulting approximation for $y(2)$.

Let $y_1 = y$ and $y_2 = y'$. Then we get the system $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$ with $\mathbf{f}(t, \mathbf{y}) = \begin{pmatrix} y_2 \\ t - y_1 - y_2 \end{pmatrix}$ and $\mathbf{y}(1) = \mathbf{y}^{(0)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. With $\mathbf{s} = \mathbf{f}(1, \mathbf{y}^{(0)}) = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ we obtain $\mathbf{y}^{(1)} = \mathbf{y}^{(0)} + h\mathbf{s} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$, so the approximation for $y(2)$ is 3 .

(b) Perform one step of the *improved Euler method* with $h = 1$ and give the resulting approximation for $y(2)$.

We get $\mathbf{s}^{(1)} = \mathbf{f}(1, \mathbf{y}^{(0)}) = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and $\mathbf{s}^{(2)} = \mathbf{f}(2, \mathbf{y}^{(0)} + h\mathbf{s}) = \mathbf{f}(2, \begin{pmatrix} 3 \\ 0 \end{pmatrix}) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and $\mathbf{y}^{(1)} = \mathbf{y}^{(0)} + \frac{h}{2}(\mathbf{s}^{(1)} + \mathbf{s}^{(2)}) = \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix}$, so the approximation for $y(2)$ is 2 .