Solution: Practice Problem for Numerical Integration

1. Consider the integral $I = \int_0^2 x^3 dx$.

(a) Find the value of the **(i)** midpoint rule, **(ii)** trapezoid rule, **(iii)** Simpson rule (on the whole interval). Find an upper bound for the error $|Q - I| \leq \cdots$ using the error formulas in each case.
Here $a = 0$, $b = 2$ and $f(x) = x^3$, so we obtain

$$Q^{\text{Midpt}} = 2 \cdot f(1) = 2, \qquad Q^{\text{Trap}} = 2 \cdot \frac{f(0) + f(2)}{2} = 8, \qquad Q^{\text{Simpson}} = 2 \cdot \frac{f(0) + 4 \cdot f(1) + f(2)}{6} = 4$$

For the error estimates we have $f''(x) = 6x$ and $\max_{[0,2]} |f''(x)| = 12$ yielding

$$\begin{aligned} \left|Q^{\text{Midpt}} - I\right| &\leq \frac{(b-a)^3}{24} \max_{[0,2]} \left|f''(x)\right| = \frac{2^3}{24} \cdot 12 = 4 \\ \left|Q^{\text{Trap}} - I\right| &\leq \frac{(b-a)^3}{12} \max_{[0,2]} \left|f''(x)\right| = \frac{2^3}{12} \cdot 12 = 8 \\ \left|Q^{\text{Simpson}} - I\right| &\leq \frac{(b-a)^5}{90 \cdot 32} \max_{[0,2]} \left|f^{(4)}(x)\right| = 0 \end{aligned}$$

since we have $f^{(4)}(x) = 0$ in this case. Recall that the Simpson rule is actually exact if $f(x)$ is a polynomial of degree≤ 3.

(b) Find the value of the composite trapezoid rule Q_2^{Trap} with 2 subintervals of equal size.
Here $N = 2$ and $h = (b-a)/N = 1$ yielding

$$Q_2^{\text{Trap}} = h \cdot \frac{[f(0) + f(1)] + [f(1) + f(2)]}{2} = 1 \cdot \frac{0 + 2 \cdot 1 + 8}{2} = 5$$

(c) Find a value N such that we can guarantee $\left|Q_N^{\text{Trap}} - I\right| \leq 10^{-10}$ for the composite trapezoid rule with N intervals of equal size.
Using $\max_{[0,2]} |f''(x)| = 12$ we obtain

$$\left|Q_N^{\text{Trap}} - I\right| \leq \frac{1}{12} \cdot \frac{(b-a)^3}{N^2} \max_{x \in [a,b]} \left|f''(x)\right| = \frac{1}{12} \cdot \frac{2^3}{N^2} \cdot 12 = \frac{8}{N^2}$$

We need to choose N such that

$$\frac{8}{N^2} \leq 10^{-10} \quad \Longleftrightarrow \quad \sqrt{\frac{8}{10^{-10}}} \leq N,$$

i.e., $N \geq 10^5 \cdot \sqrt{8} \approx 282842.7$. Therefore we need $N \geq 282843$ to guarantee $\left|Q_N^{\text{Trap}} - I\right| \leq 10^{-10}$.