1.(a) Begin with $T_1 = \bar{X}^2$ and apply δ-method

$$\bar{X}^2 = \mu^2 + 2\mu(\bar{X}-\mu) + (\bar{X}-\mu)^2$$

$$\sqrt{n}(\bar{X}^2-\mu^2) = 2\mu\sqrt{n}(\bar{X}-\mu) + \sqrt{n}(\bar{X}-\mu)(\bar{X}-\mu)$$

We know $\sqrt{n}(\bar{X}-\mu) \sim N(0,\sigma^2)$ and $\bar{X}-\mu \xrightarrow{P} 0$

so $\sqrt{n}(\bar{X}^2-\mu^2) \xrightarrow{d} N(0, 4\mu^2\sigma^2)$

$$\sqrt{n}(T_1-T_2) = \sqrt{n}(\sigma^2/n) \to 0$$

$$\sqrt{n}(T_2-T_3) = \sqrt{n}(S^2-\sigma^2)/n = (S^2-\sigma^2)/\sqrt{n} \xrightarrow{P} 0$$

because $S^2 \xrightarrow{P} \sigma^2$. Therefore, according to Slutsky's Theorem, $\sqrt{n}(T_j-\theta)$ have the same limiting distribution

(b) Now we assume $\mu^2 = \theta = 0$. Therefore the linear term in ~~the~~ the Taylor expansion vanishes.

We know $n\bar{X}^2 \sim \sigma^2\chi^2_1$ (this is an exact result)

and $E(n\bar{X}^2) = \sigma^2$. Therefore $a(n) = n$ is the required normalizing constant.

$$n(T_2-\theta) = n(\bar{X}^2-\sigma^2/n) = n\bar{X}^2-\sigma^2$$

and $n(T_2-\theta) \sim \sigma^2(\chi^2_1-1)$, a location-scale transformed χ^2_1 variable

$$n(T_2-T_3) = S^2-\sigma^2 \xrightarrow{P} 0,$$ so

$n(T_2-\theta)$ and $n(T_3-\theta)$ have the same limiting distributions

2(a) $L = \prod_{i=1}^{n}\left[e^{-\mu}\mu^{X_i}/X_i!\right] = \exp\left(n\bar{X}\log\mu - n\mu\right)/\prod_{i=1}^{n} X_i!$

$$\frac{\partial \log L}{\partial \mu} = \frac{n\bar{X}}{\mu} - n = 0 \Rightarrow \hat{\mu} = \bar{X}$$

$$\frac{\partial^2 \log L}{\partial \mu^2} = -\frac{n\bar{X}}{\mu^2} \Rightarrow \mathcal{I}_n(\mu) = \frac{n}{\mu}$$

$$-2\log\Lambda = -2n\left\{\bar{X}\log\mu_0 - \mu_0 - \bar{X}\log\bar{X} + \bar{X}\right\}$$

$$= 2n\left[\bar{X}\log(\bar{X}/\mu_0) - (\bar{X}-\mu_0)\right]$$

For large n, reject if $-2\log\Lambda > \chi^2_{1,\alpha} = z^2_{\alpha/2}$

(b) Wald test Reject iff $\dfrac{n[\hat{\mu}-\mu_0]^2}{\mathcal{I}_n^{-1}(\hat{\mu})} > z^2_{\alpha/2}$

iff $\dfrac{n[\bar{X}-\mu_0]^2}{\bar{X}/n} > z^2_{\alpha/2}$

Invert acceptance rule to get confidence interval

$L_W = \bar{X} - z_\alpha\sqrt{\bar{X}/n}$ $\quad U_W = \bar{X} + z_{\alpha/2}\sqrt{\bar{X}/n}$

Score test Reject iff $n^2(\bar{X}/\mu_0 - 1)^2/\mathcal{I}_n(\mu_0) > z^2_{\alpha/2}$

iff $\dfrac{n(\bar{X}-\mu_0)^2}{\mu_0} > z^2_{\alpha/2}$

Inverting acceptance rule means solving the quadratic inequality $n(\bar{X}-\mu)^2 \le \mu z^2_{\alpha/2}$. Roots (in μ) are

$$\bar{X} - \frac{z^2}{2n} \pm z\sqrt{\bar{X}/n}\sqrt{1+z^2/(4n\bar{X})}$$

It is straightforward to verify $L_S - L_W \to 0$, $U_S - U_W \to 0$. In fact $\sqrt{n}(L_S - L_W) \to 0$, $\sqrt{n}(U_S - U_W) \to 0$

3. Estimate θ using only X_1

(a) Method of moments: $E(X_1) = n\theta^2 \Longrightarrow \hat{\theta}_M = \sqrt{X_1/n}$

(b) Likelihood $L(\theta | X_1) = \binom{n}{X_1} \theta^{2X_1}(1-\theta^2)^{n-X_1}$

$$\Longrightarrow \hat{\theta}_M = \sqrt{X_1/n}$$

Use δ-method. Write $\eta = \theta^2$

$$\sqrt{n}\left[\sqrt{X_1/n} - \theta\right] = \sqrt{n}\left[+ \left(X_1/n - \theta^2\right)\frac{1}{2\eta^{1/2}} + R\right] \doteq \sqrt{n}\left(\frac{X_1}{n} - \theta^2\right)\frac{1}{2\theta}$$

$$\therefore \quad \sqrt{n}\left(\hat{\theta}_M - \theta\right) \xrightarrow{d} N\left(0, \frac{1-\theta^2}{4}\right) \quad \text{as } n \to \infty$$

Check asymptotic efficiency by finding Fisher info.

$$L(\theta | X_1, X_2, X_3) = \log\left(\frac{n!}{X_1!\,X_2!\,X_3!}\right) + X_1 \log\theta^2$$
$$+ X_2 \log\left(2\theta(1-\theta)\right) + X_3 \log\left((1-\theta)^2\right)$$

$$\frac{\partial \log L}{\partial \theta} = \frac{2X_1 + X_2}{\theta} - \frac{X_2 + 2X_3}{1-\theta} \qquad (*)$$

$$\frac{\partial^2 \log L}{\partial \theta^2} = -\frac{2X_1 + X_2}{\theta^2} - \frac{X_2 + 2X_3}{(1-\theta)^2}$$

$$I_n = \frac{2n}{\theta(1-\theta)}$$

Easy to see that $\frac{1-\theta^2}{4n} = \frac{(1-\theta)(1+\theta)}{4n} > \frac{\theta(1-\theta)}{2n}$

$\therefore$ $\hat{\theta}_M$ is not asymptotically efficient.

MLE of θ based on (X_1, X_2, X_3) is $\hat{\theta}^* = \frac{2X_1 + X_2}{2n}$ as can be seen from $(*)$. From ML theory

$$\sqrt{n}\left(\hat{\theta}^* - \theta\right) \xrightarrow{d} N\left(0, I_1^{-1}\right) = N\left(0, \frac{\theta(1-\theta)}{2}\right).$$

[In fact, the exact variance of $\hat{\theta}^*$ is $\theta(1-\theta)/(2n)$.]

(4) (a) $S = \sum_{i=1}^{n} I\{X_i > 0\}$ is binomial $(n, 1-F(0))$

$$\therefore \frac{S - n(1-F(0))}{\sqrt{nF(0)(1-F(0))}} \xrightarrow{d} N(0,1). \quad (*)$$

Under H_0, $F(0) = 1/2$ because 0 is the median

$$\therefore \quad c_{n\alpha} \doteq \frac{n}{2} + z_\alpha \sqrt{n/4}$$

(b) If ~~[illegible]~~ $\theta = 1$, $F(1) = \frac{1}{2}$ and $F(0) < \frac{1}{2}$

$$P\left[S > \frac{n}{2} + z_\alpha \sqrt{n/4}\right]$$

$$= P\left[\frac{S - n(1-F(0))}{\sqrt{nF(0)(1-F(0))}} > \frac{n/2 - n + nF(0)}{\sqrt{nF(0)(1-F(0))}} + z_\alpha \sqrt{\frac{1/4}{F(0)(1-F(0))}}\right]$$

$$\doteq P\left[Z > -\sqrt{n}\frac{1/2 - F(0)}{\sqrt{F(0)(1-F(0))}} + \frac{z_\alpha}{2}\frac{1}{\sqrt{F(0)(1-F(0))}}\right]$$

where $Z \sim N(0,1)$.

Note that the power depends only on $F(0) < \frac{1}{2}$ and not on θ.

(c) Let $n \to \infty$. Then $-\sqrt{n}\dfrac{1/2 - F(0)}{\sqrt{F(0)(1-F(0))}} \to -\infty$

and $\dfrac{z_\alpha}{2}\dfrac{1}{\sqrt{F(0)(1-F(0))}}$ is fixed.

$\therefore$ power $\longrightarrow 1$