NOTES ON THE SPECTRAL RESOLUTION OF A SELF-ADJOINT OPERATOR

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Contents

Introduction

In [6], John von Neumann created the mathematical foundation of quantum mechanics: to do so he formulated the theory of general Hilbert spaces and of the linear operators, both bounded and unbounded, on such spaces. A centerpiece of this theory is the existence of a spectral resolution for a self-adjoint operator $T\colon D \subseteq H \to H$ acting on a complex Hilbert space H, that is, of the following representation of such an operator as the Stieltjes integral with respect to an increasing path $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ of orthogonal projections on H:

$$Th = \int_{-\infty}^{\infty} \lambda \, d\mathcal{E}_\lambda h \text{ for all } h \text{ in } D. \tag{1}$$

He "sketched in broad outline" ([6, p. 154]) his method of proof, which proceeded as follows: he first established a spectral resolution for a bounded unitary operator, that is, for a bounded operator $T\colon H \to H$ for which $T^* = T^{-1}$, and then for a general self-adjoint operators $T\colon D \subseteq H \to H$ by passing to its Cayley transform $(T + \imath \cdot \mathrm{I})(T - \imath \cdot \mathrm{I})^{-1}$, which is unitary.

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Since then many different proofs and formulations of the existence of a spectral resolution have been given. We mention two of these: In [2], Nelson Dunford and Jacob Schwartz first used Gelfand's theory of B^* algebras to establish a spectral resolution for a bounded linear operator that is normal, that is, for a bounded linear operator T for which $T^*T = TT^*$; and then for a general self-adjoint operator T, by passing, via the Riesz-Dunford resolvent calculus, to the operator $(T - z \cdot \mathrm{I})^{-1}$, which is normal; In [4], Peter Lax, following Doob and Koopman [1], proved a representation theorem of Herzglotz-Riesz for the boundary values of a function that is analytic on the upper half-plane, and a theorem of Nevinlinna regarding the growth of analytic functions with positive imaginary part. He used these two results to examine the behavior, for a self-adjoint operator T, of the resolvent function $z \mapsto (T - z \cdot \mathrm{I})^{-1}$, which is analytic in the upper half-plane, and thereby proved the existence of a spectral resolution of T.

Paul Halmos observed that the spectral resolution, even for bounded symmetric operators, is "widely regarded as mysterious and deep"([3, p. 241]) and often is viewed as lying only in the domain of the specialist. One reason for this is that, as von Neumann already noted ([6, p. 119]), the integral representation (1) is not easily recognizable as being related to the eigenvalue problem $Th = \lambda h$. Moreover, frequently, as in [2] and [4], the resolution is expressed not as a Stieltjes integral with respect to a path of orthogonal projections but as an integral with respect to a projection-valued measure on the Borel sets of the spectrum of T, which is quite far from the eigenvalue problem $Th = \lambda h$. A further reason is that, in many comprehensive texts (see, for instance, [2], [4], [10]), a proof of the existence of a spectral resolution is given by relying on material that spans the preceding several hundred pages, making it difficult to discern what comprises the essential ingredients of a particular proof. Finally, we note that in many presentations either extra assumptions are made (for instance, separability of H in [7]) or uniqueness is not addressed (for instance, in [4]).

Our goal here is to address the above points. We present a direct, fully detailed and self-contained, proof of the existence of a unique spectral resolution for an unbounded self-adjoint operator T: the concept of spectral resolution is precisely the original one defined by von Neumann ([6, p. 118]). This proof is based on the equivalence of the integral representation (1) for T to a property of the eigenvalue problem $Th = \lambda h$. More precisely, let $\{\mathcal{E}_\lambda\}_{\lambda \in \mathbf{R}}$ be a right-continuous, increasing path of orthogonal projections on H with the property that $h = \int_{-\infty}^{\infty} d\mathcal{E}_\lambda h$ for all h in H. Let $T \colon D \subseteq H \to H$ be self-adjoint. We prove that

$$D = \left\{ h \text{ in } H \;\middle|\; \int_{-\infty}^{\infty} \lambda^2 \, d\langle \mathcal{E}_\lambda h, h\rangle < \infty \right\} \quad \text{and} \quad Th = \int_{-\infty}^{\infty} \lambda \, d\mathcal{E}_\lambda h \text{ for all } h \text{ in } D \tag{2}$$

if and only if for $\alpha < \beta$, T maps $\mathcal{E}_\beta(H) \cap [\mathcal{E}_\alpha(H)]^\perp$ into itself and

$$\|Th - \lambda_0 h\| \leq (\beta - \alpha)\|h\| \text{ for all } h \text{ in } \mathcal{E}_\beta(H) \cap [\mathcal{E}_\alpha(H)]^\perp \text{ and } \lambda_0 \text{ in } [\alpha, \beta]. \tag{3}$$

In section 1.1, we establish elementary algebraic properties of orthogonal projections and elementary results regarding Stieltjes integration of a continuous real-valued function with respect to a path of orthogonal projections, based on which we prove the equivalence of (2) and (3) in the case T in $\mathcal{L}(H)$ symmetric. The following section is devoted to constructing a functional calculus for $f(T)$, for T in $\mathcal{L}(H)$ symmetric and f is a bounded, real-valued Borel function on the spectrum of T. For completeness and clarity, the entirety of the background in spectral theory for bounded symmetric operators that we need to construct this calculus is established from scratch in the brief subsection 1.3. In subsection 1.4, we prove that, for T in $\mathcal{L}(H)$ symmetric, if we define, for λ in $\mathbf{R}$, $\mathcal{E}_\lambda \equiv f_\lambda(T)$, where f_λ is the characteristic function of the interval $(-\infty, \lambda]$, then $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ is the unique right-continuous, increasing path of orthogonal projections with the property that $h = \int_{-\infty}^{\infty} \lambda d\mathcal{E}_\lambda h$ for all h in H for which (3), and hence (2), holds.

In Section 2, we turn to the proof of the existence of a unique spectral resolution for an unbounded self-adjoint operator. We follow the approach of using bounded approximations, as in [10]. We observe that a few points in [10] need touching up (for instance, the actual definition of the integral on the right-hand side of (2) (see [10,]), and the lower value of the integrand in the case of bounded operators (see the footnote on [,]). By first proving the equivalence of (2) and (3) these points are clarified. Moreover, of course, this equivalence also explicitly reveals the relationship between a spectral resolution of T and the eigenvalue problem $Th = \lambda h$. The proof is based on the approximation of such an operator by a sequence of bounded symmetric operators. Indeed, let $T\colon D \subseteq H \to H$ be self-adjoint. If $\{H_n\}$ is an ascending sequence of closed subspaces of H for which $\cup_{n=1}^{\infty} H_n$ is a dense subset of D and T maps each H_n into itself, we call the sequence $\{T\colon H_n \to H_n\}_{n=1}^{\infty}$ a sequence of bounded approximations of $T\colon D \subseteq H \to H$. A theorem of Hellinger and Toeplitz tells us that each approximant $T\colon H_n \to H_n$ is a bounded symmetric operator. Work of Lorch, Nagy and Riesz (see [9] and [10]) may be synthesized in what we call the Bounded Approximation Theorem: every self-adjoint operator $T\colon D \subseteq H \to H$ has a sequence of bounded approximations $\{T\colon H_n \to H_n\}_{n=1}^{\infty}$, and, for any such sequence, if, for each n, Q_n is the orthogonal projection of H onto H_n, the domain D comprises those h in H for which $\{TQ_nh\}_{n=1}^{\infty}$ is bounded and for each such h, $Th = \lim_{n\to\infty} TQ_nh$. We postpone a detailed proof of this theorem until the final section. By passage to the limit as $n \to \infty$, we first deduce the equivalence of (2) and (3) for a self-adjoint operator $T\colon D \subset H \to H$ from the equivalence of these for each approximant, and then deduce that (3) holds since it holds for each approximant. We establish uniqueness by taking the limit for a special approximating sequence.

A remark regarding the origins of the spectral resolutions of a self-adjoint operator is in order. For a symmetric operator T in $\mathcal{L}(H)$, the origins of this development lie in the seminal work of David Hilbert and his student Erhard Schmidt on eigenvalue problems for integral equations. John von Neumann [5], Frederick Riesz [8] and Marshall Stone [12] were the first to consider the spectral resolution of a self-adoint operator $T\colon D \subseteq H \to H$.

Throughout, H denotes a complex Hilbert space, equipped with a Hermitian inner-product $\langle\cdot,\,\cdot\rangle$. The orthogonal complement of a subspace V of H is denoted by $V^\perp$. The space of bounded linear operators on H is denoted by $\mathcal{L}(H)$. An operator T in $\mathcal{L}(H)$ is said to be invertible provided it is is one-to-one and onto: the Open Mapping Theorem tells us that the inverse is bounded. For T in $\mathcal{L}(H)$, the resolvent of T, $\rho(T)$, is defined to be the set of complex numbers λ for which $\lambda\cdot\mathrm{I} - T$ is invertible. The spectrum of T, $\sigma(T)$, is the complement, in the complex numbers, of $\rho(T)$. An operator T in $\mathcal{L}(H)$ is said to be symmetric provided

$$\langle Tu,\, v\rangle = \langle u,\, Tv\rangle \text{ for all } u, v \text{ in } H.$$

1. The Spectral Resolution of a Bounded Symmetric Operator

1.1. **Paths of Orthogonal Projections and Stieltjes Integration.** There is a natural ordering among symmetric operators in $\mathcal{L}(H)$. For two such operators T and S, we write $T \leq S$ provided

$$\langle Th, h\rangle \leq \langle Sh, h\rangle \text{ for all } h \text{ in } H.$$

An operator P in $\mathcal{L}(H)$ is said to be a projection provided $P^2 = P$. A projection P is said to be orthogonal provided $(\mathrm{I} - P)H$ is the orthogonal complement of $P(H)$, which is equivalent to the assertion that it is symmetric.

Lemma 1. *Let P and Q be orthogonal projections on H for which $P \leq Q$. Then*

$$PQ = QP = P, \text{ and} \tag{4}$$

$$Q - P \text{ is the orthogonal projection of } H \text{ onto } Q(H) \cap P(H)^{\perp}. \tag{5}$$

Moreover, if R is an orthogonal projection on H for which $Q \leq R$, then

$$\text{the spaces } (Q-P)H \text{ and } (R-Q)H \text{ are orthogonal.} \tag{6}$$

If $T\colon Q(H) \to Q(H)$ is symmetric and $T(P(H)) \subseteq P(H)$, then

$$\|TQh\|^2 - \|TPh\|^2 = \|T(Q-P)h\|^2 \text{ for all } h \text{ in } Q(H). \tag{7}$$

Proof. Let u belong to $P(H)$. Then

$$\langle u, u\rangle = \langle Pu, u\rangle \leq \langle Qu, u\rangle = \langle u, u\rangle - \langle (\mathrm{I} - Q)u, (\mathrm{I} - Q)u\rangle.$$

Thus $\langle (\mathrm{I} - Q)u, (\mathrm{I} - Q)u\rangle = 0$ and hence $u = Qu$. Therefore $P(H) \subseteq Q(H)$, from which we deduce that $QP = P$. To verify (4) it remains to show that $PQ = P$. Indeed, for u, v in H, since P and Q are symmetric,

$$\langle PQu, v\rangle = \langle u, QPv\rangle = \langle u, Pv\rangle = \langle Pu, v\rangle,$$

that is, $PQ = P$. To verify (5), first observe that since P and Q commute, $Q - P$ is a projection, and is an orthogonal projection since it is symmetric. Moreover, $Q - P$ is the identity on $Q(H) \cap P(H)^{\perp}$ and maps H into $Q(H) \cap P(H)^{\perp}$. Hence (5) holds. The relations $Q \leq R$ and $P \leq R$ may be substituted in the above arguments for the relation $P \leq Q$, and hence $QR = Q$ and $PR = P$. Therefore

$$(Q-P)(R-Q) = QR - QQ - PR + PP = 0,$$

which is equivalent to assertion (6). To verify (7), observe that since T is symmetric and maps $Q(H)$ into itself and $P(H)$ into itself, it maps $Q(H) \cap P(H)^{\perp}$ into itself. According to (5), $(Q-P)h$ belongs to $Q(H) \cap P(H)^{\perp}$. Thus $T(Ph)$ and $T(Q-P)h$ are orthogonal. Since $QP = P$ and

$$TQh = TQPh + TQ(h - Ph) = TPh + T(Q-P)h,$$

and therefore we obtain (7). □

Consider a path $\{\mathcal{E}_\lambda\}_{\lambda \in \mathbf{R}}$ of orthogonal projections on H that is increasing in the sense that if $\alpha \leq \beta$, then $\mathcal{E}_\alpha \leq \mathcal{E}_\beta$. Conclusion (4) of the preceding lemma tells that for each α, β

$$\mathcal{E}_\alpha \circ \mathcal{E}_\beta = \mathcal{E}_\beta \circ \mathcal{E}_\alpha = \mathcal{E}_{\min\{\alpha,\beta\}}, \tag{8}$$

so that, in particular, $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ is a family of commuting operators. According to (5), for $\alpha < \beta$,

(9) $\quad \mathcal{E}_\beta - \mathcal{E}_\alpha$ is the orthogonal projection of H onto $(\mathcal{E}_\beta - \mathcal{E}_\alpha)H = \mathcal{E}_\beta(H) \cap [\mathcal{E}_\alpha(H)]^\perp$.

For each λ and $\alpha \le \beta$, since $\mathcal{E}_\lambda$ commutes with $\mathcal{E}_\beta - \mathcal{E}_\alpha$, it maps $(\mathcal{E}_\beta - \mathcal{E}_\alpha)H$ into itself and hence

$$\mathcal{E}_\lambda \text{ maps } \mathcal{E}_\beta(H) \cap [\mathcal{E}_\alpha(H)]^\perp \text{ into itself.} \tag{10}$$

Finally, conclusion (6) of this lemma tells us that, for $\alpha = \lambda_0 < \lambda_1 < \ldots < \lambda_n = \beta$, there in the following direct sum orthogonal decomposition:

$$(\mathcal{E}_\beta - \mathcal{E}_\alpha)H = (\mathcal{E}_{\lambda_1} - \mathcal{E}_{\lambda_0})H \oplus (\mathcal{E}_{\lambda_2} - \mathcal{E}_{\lambda_1})H \oplus \cdots \oplus (\mathcal{E}_{\lambda_n} - \mathcal{E}_{\lambda_{n-1}})H, \tag{11}$$

so that if $h = \sum_{k=1}^n h_k$, with each h_k in $(\mathcal{E}_{\lambda_k} - \mathcal{E}_{\lambda_{k-1}})H$, then

$$\|h\|^2 = \sum_{k=1}^n \|h_k\|^2. \tag{12}$$

Let $[a,\, b]$ be a compact interval and V a normed linear space. Consider a path $g\colon [a,\, b] \to V$ and a function $f\colon [a,\, b] \to \mathbf{R}$. We recall what it means for f to be Stieltjes integrable with respect to g over $[a, b]$, and, when it is integrable, the value of the integral.

We call $\pi = \{\lambda_k\}_{k=0}^n$ a partition of $[a,\, b]$ provided $a = \lambda_0 < \lambda_1 < \ldots < \lambda_{n-1} < \lambda_n = b$. The gap of π, which is denoted by gap(π), is defined to be the maximum of $\{\lambda_k - \lambda_{k-1}\}_{1\le k\le n}$. A set $c = \{c_1, c_2, \ldots, c_n\}$ is called a choice set for π provided that, for $1 \le k \le n$, c_k belongs to $[\lambda_{k-1}, \lambda_k]$. We call

$$\mathrm{Sum}(f,\, g,\, \pi, c) \equiv \sum_{k=0}^n f(c_k)(g(\lambda_k) - g(\lambda_{k-1}))$$

a Stieltjes sum for f with respect to g over $[a, b]$. Suppose there is a vector v in V with the following property: for each $\epsilon > 0$, there is a $\delta > 0$, such that, if π is any partition of $[a,\, b]$ and c any choice set for π, then

$$\|v - \mathrm{Sum}(f,\, g,\, \pi, c)\| < \epsilon \text{ if } \mathrm{gap}(\pi) < \delta.$$

There can be at most one such vector. If there is such a vector v we say that f is Stieltjes integrable with respect to g over $[a, b]$ and write

$$v = \int_a^b f(\lambda) \cdot\, dg(\lambda).$$

Integration over infinite intervals is defined for a path $g\colon (-\infty,\, \infty) \to V$ and a function $f\colon (-\infty,\, \infty) \to \mathbf{R}$ by

$$\int_{-\infty}^{\infty} f(\lambda) \cdot\, dg(\lambda) \equiv \lim_{\substack{\gamma\to-\infty \\ \eta\to\infty}} \int_\gamma^\eta f(\lambda) \cdot\, dg(\lambda)$$

provided this limit exists.

We are interested in the cases $V = \mathbf{R}$, $V = H$ and $V = \mathcal{L}(H)$. Observe that if f is Stieltjes integrable with respect to $\mathcal{E}\colon [a, b] \to \mathcal{L}(H)$ over $[a, b]$ and has Stieltjes integral T, then, for all h in H,

$$Th = \int_a^b f(\lambda) \cdot d\mathcal{E}_\lambda h \quad \text{and} \quad \langle Th, h\rangle = \int_a^b f(\lambda) \cdot d\langle \mathcal{E}_\lambda h, h\rangle.$$

All of the properties of Stieltjes integration that we need are collected below.

Proposition 2. *Let $\{\mathcal{E}_\lambda\}_{\lambda \in [a, b]}$ be an increasing path of orthogonal projections on H and the function $f\colon [a, b] \to \mathbf{R}$ be continuous. Then f is Stieltjes integrable with respect to $\mathcal{E}$ over $[a, b]$ and the Stieltjes integral $T\colon H \to H$ is a bounded symmetric operator. Moreover, for all in H,*

$$\|Th\|^2 = \int_a^b f^2(\lambda) \cdot d\langle \mathcal{E}_\lambda h, h\rangle, \tag{13}$$

and for $[\alpha, \beta] \subseteq [a, b]$,

$$\int_a^b f(\lambda) \cdot d\,\mathcal{E}_\lambda(\mathcal{E}_\beta - \mathcal{E}_\alpha)h = \int_\alpha^\beta f(\lambda) \cdot d\,\mathcal{E}_\lambda h. \tag{14}$$

Proof. We deduce from the completeness of $\mathcal{L}(H)$ that to verify integrability it is necessary and sufficient to verify the Cauchy integrability criterion, namely, to show that for each $\epsilon > 0$, there is a $\delta > 0$ such that

$$\|\operatorname{Sum}(f, \mathcal{E}, \pi, c) - \operatorname{Sum}(f, \mathcal{E}, \pi', c')\| < \epsilon \text{ if } \operatorname{gap}(\pi) < \delta \text{ and } \operatorname{gap}(\pi') < \delta.$$

Let $\epsilon > 0$. By the uniform continuity of f on $[a, b]$, we can choose $\delta > 0$ such that for s, t in $[a, b]$, $|f(s) - f(t)| < \epsilon$ if $|s - t| < \delta$. We claim that this δ responds to the ϵ challenge with respect to verifying the Cauchy integrability criterion. Indeed, let π and π' be partitions of $[a, b]$ for which $\operatorname{gap}(\pi) < \delta$ and $\operatorname{gap}(\pi') < \delta$.

First consider the special case $\pi = \pi'$. Let $c = \{c_1, c_2, \ldots, c_n\}$ and $c' = \{c'_1, c'_2, \ldots, c'_n\}$ be choice sets for π. Let h belong to H. We have

$$[\operatorname{Sum}(f, \mathcal{E}, \pi, c) - \operatorname{Sum}(f, \mathcal{E}, \pi, c')]h = \sum_{k=1}^n (f(c_k) - f(c'_k))(\mathcal{E}_{\lambda_k} - \mathcal{E}_{\lambda_{k-1}})h.$$

We may therefore appeal to (12) to deduce that

$$\begin{aligned}
\|[\operatorname{Sum}(f, \mathcal{E}, \pi, c) - \operatorname{Sum}(f, \mathcal{E}, \pi, c')]h\|^2 &= \sum_{k=1}^n |f(c_k) - f(c'_k)|^2 \, \|(\mathcal{E}_{\lambda_k} - \mathcal{E}_{\lambda_{k-1}})h\|^2 \\
&\leq \epsilon^2 \sum_{k=1}^n \|(\mathcal{E}_{\lambda_k} - \mathcal{E}_{\lambda_{k-1}})h\|^2 \\
&= \epsilon^2 \, \|(\mathcal{E}_b - \mathcal{E}_a)h\|^2 \\
&\leq \epsilon^2 \, \|h\|^2 \text{ for all } h \text{ in } H.
\end{aligned}$$

Thus $\|\operatorname{Sum}(f, \mathcal{E}, \pi, c) - \operatorname{Sum}(f, \mathcal{E}, \pi, c')\| < \epsilon$, and so this choice of δ responds to the ϵ challenge when the partitions are equal.

In the case that $\pi \neq \pi'$ we argue as follows. Let $\pi^* = \{\beta_k\}_{k=0}^m$ be a partition of $[a, b]$ that is a refinement of both π and π'. Let c^* be the unique choice set for π^* with the property that if the i-th interval induced by the partition π^* is contained in the j-th interval induced by the partition of π, then $c_i^* = c_j$. Let c'^* be the unique choice set for π^* that is similarly related to π'. Observe that

$$\mathrm{Sum}(f,\, \mathcal{E},\, \pi,\, c) = \mathrm{Sum}(f,\, \mathcal{E},\, \pi^*,\, c^*) \text{ and } \mathrm{Sum}(f,\, \mathcal{E},\, \pi',c') = \mathrm{Sum}(f,\, \mathcal{E},\, \pi'^*,\, c'^*).$$

But $\mathrm{gap}(\pi^*) < \delta$ and therefore, as we argued above, $|\mathrm{Sum}(f,\, \mathcal{E},\, \pi^*,\, c^*) - \mathrm{Sum}(f,\, \mathcal{E},\, \pi'^*,\, c'^*)| < \epsilon$, so that $|\mathrm{Sum}(f,\, \mathcal{E},\, \pi,\, c) - \mathrm{Sum}(f,\, \mathcal{E},\, \pi',\, c')| < \epsilon$. This completes the proof of the integrability of f with respect to $\mathcal{E}\colon [a,\, b] \to \mathcal{L}(H)$ over $[a,\, b]$.

Each Stieltjes sum is a bounded symmetric operator and hence so is their limit in $\mathcal{L}(H)$, the integral, which we denote by T. It remains to verify (13) and (14). Let h belong to H. For the Stieltjes sum $\mathrm{Sum}(f,\, \mathcal{E},\, \pi,\, c) = \sum_{k=1}^{n} f(c_k)(\mathcal{E}_{\lambda_k} - \mathcal{E}_{\lambda_{k-1}})$, we deduce from (11) that

$$\langle \mathrm{Sum}(f,\, \mathcal{E},\, \pi,\, c)h,\, \mathrm{Sum}(f,\, \mathcal{E},\, \pi,\, c)h\rangle = \sum_{k=1}^{n} f^2(c_k)[\langle \mathcal{E}_{\lambda_k}h,\, h\rangle - \langle \mathcal{E}_{\lambda_{k-1}}h,\, h\rangle].$$

Since f and f^2 are integrable with respect to $\mathcal{E}\colon [a,\, b] \to \mathcal{L}(H)$ over $[a,\, b]$, we take the limit as $\mathrm{gap}(\pi) \to 0$ to deduce that $\|Th\|^2 = \langle Th,\, Th\rangle = \int_\alpha^\beta f^2(\lambda) \cdot d\,\langle \mathcal{E}_\lambda h,\, h\rangle$. So (13) is verified.

We deduce from (8) that the path $\lambda \mapsto \mathcal{E}_\lambda(\mathcal{E}_\beta - \mathcal{E}_\alpha)$ takes the constant value 0 on the interval $(-\infty,\, \alpha]$ and the constant value $\mathcal{E}_\beta - \mathcal{E}_\alpha$ on the interval $[\beta,\, \infty)$, so that

$$\int_a^b f(\lambda) \cdot d\,\mathcal{E}_\lambda(\mathcal{E}_\beta - \mathcal{E}_\alpha)h = \int_\alpha^\beta f(\lambda) \cdot d\,\mathcal{E}_\lambda(\mathcal{E}_\beta - \mathcal{E}_\alpha)h.$$

On the other hand, again by (8), the path $\lambda \mapsto \mathcal{E}_\lambda(\mathcal{E}_\beta - \mathcal{E}_\alpha) - \mathcal{E}_\lambda$ takes the constant value $-\mathcal{E}_\alpha$ on $[\alpha,\, \beta]$, and consequently

$$\int_\alpha^\beta f(\lambda) \cdot d\,\mathcal{E}_\lambda(\mathcal{E}_\beta - \mathcal{E}_\alpha)h = \int_\alpha^\beta f(\lambda) \cdot d\,\mathcal{E}_\lambda h.$$

So (14) is verified.

□

1.2. The Spectral Resolution and Estimates of $Th - \lambda h$.

Definition. *If T is a bounded symmetric operator on H and h belongs to H, by symmetry, $\langle Th,\, h\rangle = \langle h,\, Th\rangle$ and, since $\langle \cdot,\, \cdot\rangle$ is an Hermitian inner product, $\langle h,\, Th\rangle$ is the complex conjugate of $\langle Th,\, h\rangle$. Therefore $\langle Th,\, h\rangle$ is real. Define*

$$m(T) \equiv \inf_{h\in H, h\neq 0} \frac{\langle Th,\, h\rangle}{\langle h,\, h\rangle} \quad \text{and} \quad M(T) \equiv \sup_{h\in H, h\neq 0} \frac{\langle Th,\, h\rangle}{\langle h,\, h\rangle},$$

and call $m = m(T)$ and $M = M(T)$ the **spectral bounds** *for T.*

Definition. *We call an increasing path of orthogonal projections $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$* **right-continuous** *provided for each h in H and real number λ_0, $\lim_{\lambda\to\lambda_0^+} \mathcal{E}_\lambda h = \mathcal{E}_{\lambda_0} h$.*

Lemma 3. *Let T in $\mathcal{L}(H)$ be a symmetric operator that has spectral bounds m and M. Let $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ be a right-continuous, increasing path of orthogonal projections on H with the property that $h = \int_{-\infty}^{\infty} d\,\mathcal{E}_\lambda h$ for all h in H. Assume that for $\alpha < \beta$,*

$$\|Th - \lambda_0 h\| \leq (\beta - \alpha)\|h\| \text{ for all } h \text{ in } \mathcal{E}_\beta(H) \cap [\mathcal{E}_\alpha(H)]^\perp \text{ and } \lambda_0 \text{ in } [\alpha, \beta]. \tag{15}$$

Then $\mathcal{E}_M = \mathrm{I}$ *and if* $m' < m$, $\mathcal{E}_{m'} = 0$.

Proof. We deduce from (15) and the Cauchy-Schwarz Inequality that for $\alpha < \beta$ and $\alpha \leq \lambda_0 \leq \beta$,

$$-(\beta-\alpha)\langle h,\, h\rangle \leq \langle Th - \lambda_0 h,\, h\rangle \leq (\beta-\alpha)\langle h,\, h\rangle \text{ for all } h \text{ in } \mathcal{E}_\beta(H) \cap [\mathcal{E}_\alpha(H)]^\perp.$$

Let $m' < m$. Choose α and β such that $\alpha < m' < \beta$ and $\beta - \alpha < m - m'$. Taking $\lambda_0 = m'$ in the preceding inequalities and using the definition of m, we have that

$$m\langle h,\, h\rangle \leq \langle Th,\, h\rangle \leq [m' + \beta - \alpha]\langle h,\, h\rangle \text{ for all } h \text{ in } \mathcal{E}_\beta(H) \cap [\mathcal{E}_\alpha(H)]^\perp.$$

But $m > m' + \beta - \alpha$. Therefore $\mathcal{E}_\beta(H) \cap [\mathcal{E}_\alpha(H)]^\perp = \{0\}$. According to (9), $\mathcal{E}_\beta(H) \cap [\mathcal{E}_\alpha(H)]^\perp = (\mathcal{E}_\beta - \mathcal{E}_\alpha)H$. Thus the path $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ is constant in a neighborhood of m'. We argue, using the connectedness of $(-\infty, m)$, that $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ takes a constant value P on $(-\infty, m)$. A similar argument tells us that that $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ takes a constant value Q on (M, ∞) and hence, by right-continuity, on $[M, \infty)$ But $\mathrm{I} = \int_{-\infty}^{\infty} d\,\mathcal{E}_\lambda = Q - P$. Since P and Q are orthogonal projections, $Q = \mathrm{I}$ and $P = 0$. □

Theorem 4. *Let* $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ *be a right-continuous, increasing path of orthogonal projections on* H *with the property that* $h = \int_{-\infty}^{\infty} d\mathcal{E}_\lambda h$ *for all* h *in* H. *Let* T *in* $\mathcal{L}(H)$ *be symmetric. Then*

$$Th = \int_{-\infty}^{\infty} \lambda\, d\mathcal{E}_\lambda h \textit{ for all } h \textit{ in } H \tag{16}$$

if and only if T *maps each* $\mathcal{E}_\lambda(H)$ *into itself and, for* $\alpha < \beta$,

$$\|Th - \lambda_0 h\| \leq (\beta - \alpha)\|h\| \textit{ for all } h \textit{ in } \mathcal{E}_\beta(H) \cap [\mathcal{E}_\alpha(H)]^\perp \textit{ and } \lambda_0 \textit{ in } [\alpha, \beta]. \tag{17}$$

Proof. First assume (16). We deduce from (8) that the $\mathcal{E}_\lambda$'s commute and so each $\mathcal{E}_\lambda$ commutes with each Stieltjes sum associated with the integral in (16) and consequently also commutes with the integral, T. Hence, T maps each $\mathcal{E}_\lambda(H)$ into itself. Choose $\alpha < \beta$. Let λ_0 belong to $[\alpha, \beta]$ and h belong to $\mathcal{E}_\beta(H) \cap [\mathcal{E}_\alpha(H)]^\perp$. Then, by (9), $h = (\mathcal{E}_\beta - \mathcal{E}_\alpha)h$ so that

$$\lambda_0\, h = \lambda_0\, (\mathcal{E}_\beta - \mathcal{E}_\alpha)h = \int_\alpha^\beta \lambda_0\, d\mathcal{E}_\lambda h.$$

and furthermore, by (14),

$$Th = \int_{-\infty}^{\infty} \lambda\, d\mathcal{E}_\lambda h = \int_\alpha^\beta \lambda\, d\mathcal{E}_\lambda h.$$

Substitute $\lambda - \lambda_0$ for $f(\lambda)$ in (13) to obtain, since $\langle \mathcal{E}_\beta h,\, h\rangle - \langle \mathcal{E}_\alpha h,\, h\rangle = \|(\mathcal{E}_\beta - \mathcal{E}_\alpha)h\|^2$,

$$\|Th - \lambda_0 h\|^2 = \int_\alpha^\beta (\lambda - \lambda_0)^2 d\langle \mathcal{E}_\lambda h,\, h\rangle \leq (\beta-\alpha)^2\|(\mathcal{E}_\beta - \mathcal{E}_\alpha)h\|^2 = (\beta-\alpha)^2\|h\|^2.$$

Therefore (16) implies (17).

Now assume T maps each $\mathcal{E}_\lambda(H)$ into itself and (17) holds. Let m and M be the spectral bounds for T. Choose $a < m$ and set $b = M$. The preceding lemma tells us that $\mathcal{E}_a = 0$ and $\mathcal{E}_b = \mathrm{I}$. Let $\pi = \{\lambda_k\}_{k=0}^n$ be a partition of $[a,\, b]$ and c a choice set for this partition. Define $f(\lambda) \equiv \lambda$. Since $\mathcal{E}_a = 0$ and $\mathcal{E}_b = \mathrm{I}$,

$$\mathrm{I} = \sum_{k=1}^n (\mathcal{E}_{\lambda_k} - \mathcal{E}_{\lambda_{k-1}}),$$

and therefore

$$T - S(f,\, \mathcal{E},\, \pi,\, c) = \sum_{k=1}^{n} (T - c_k \cdot \mathrm{I}) \circ (\mathcal{E}_{\lambda_k} - \mathcal{E}_{\lambda_{k-1}}).$$

By assumption, T maps each $\mathcal{E}_\lambda(H)$ into itself, and hence, since T is symmetric, commutes with $\mathcal{E}_\lambda$. Therefore, for $\alpha < \beta$, T commutes with the projection $\mathcal{E}_\beta - \mathcal{E}_\alpha$ and so maps $[\mathcal{E}_\beta - \mathcal{E}_\alpha]H$ into itself. In particular, for $1 \le k \le n$, T maps each $[\mathcal{E}_{\lambda_k} - \mathcal{E}_{\lambda_{k-1}}]H$ into itself. Let h belong to H. We appeal to (11) to deduce that

$$\|Th - S(f,\, \mathcal{E},\, \pi,\, c)h\|^2 = \sum_{k=1}^{n} \|(T - c_k \cdot \mathrm{I}) \circ (\mathcal{E}_{\lambda_k} - \mathcal{E}_{\lambda_{k-1}})h\|^2.$$

For $1 \le k \le n$, since, by (5), $(\mathcal{E}_{\lambda_k} - \mathcal{E}_{\lambda_{k-1}})h$ belongs to $\mathcal{E}_{\lambda_k}(H) \cap [\mathcal{E}_{\lambda_{k-1}}(H)]^\perp$, we may invoke assumption (17) with $[\alpha, \beta]$ substituted by $[\lambda_{k-1}, \lambda_k]$ and h substituted by $(\mathcal{E}_{\lambda_k} - \mathcal{E}_{\lambda_{k-1}})\, h$, to deduce that

$$\begin{aligned} \|Th - \mathrm{Sum}(f,\, \mathcal{E},\, \pi,\, c)h\|^2 &\le \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1})^2 \|(\mathcal{E}_{\lambda_k} - \mathcal{E}_{\lambda_{k-1}})\, h\|^2 \\ &\le \mathrm{gap}(\pi)^2 \sum_{k=1}^{n} \|(\mathcal{E}_{\lambda_k} - \mathcal{E}_{\lambda_{k-1}})\, h\|^2 \\ &= \mathrm{gap}(\pi)^2 \|(\mathcal{E}_\beta - \mathcal{E}_\alpha)h\|^2 \\ &\le \mathrm{gap}(\pi)^2\, \|h\|^2. \end{aligned}$$

Therefore

$$\|T - \mathrm{Sum}(f,\, \mathcal{E},\, \pi,\, c)\| \le \mathrm{gap}(\pi).$$

Thus $f(\lambda) \equiv \lambda$ is Stieltjes integrable with respect to $\{\mathcal{E}_\lambda\}_{\lambda \in \mathbf{R}}$ over $[a,\, b]$, and its Stieltjes integral is T. Since $\{\mathcal{E}_\lambda\}_{\lambda \in \mathbf{R}}$ takes the constant value 0 on the interval $(-\infty,\, a]$ and the constant value I on the interval $[b,\, \infty)$, the integral over $[a,\, b]$ equals the integral over $(-\infty,\, \infty)$. $\square$

1.3. **The Spectral Mapping, Boundary and Radius Theorems,** For a polynomial $p(t) = \sum_{k=0}^{n} a_k t^k$ and operator T in $\mathcal{L}(H)$, we define $p(T) \equiv \sum_{k=0}^{n} a_k T^k$, where $T^0 \equiv \mathrm{I}$.

Theorem 5 (The Spectral Mapping Theorem)**.** *Let the operator T belong to $\mathcal{L}(H)$ and p be a polynomial. Then*

$$\sigma(p(T)) = p(\sigma(T)). \tag{18}$$

Proof. First suppose λ_0 belongs to $\sigma(T)$. Now λ_0 is a root of the polynomial $p(\lambda) - p(\lambda_0)$. Therefore there is a polynomial q for which $p(\lambda) - p(\lambda_0) = (\lambda - \lambda_0)q(\lambda)$. Consequently,

$$p(T) - p(\lambda_0) \cdot \mathrm{I} = (T - \lambda_0 \cdot \mathrm{I})q(T) = q(T)(T - \lambda_0 \cdot \mathrm{I}).$$

Since $T - \lambda_0 \cdot \mathrm{I}$ either fails to be one-to-one or fails to be onto, the operator $p(T) - p(\lambda_0) \cdot \mathrm{I}$ has the same property. Thus $p(\lambda_0)$ belongs to $\sigma(p(T))$.

Now assume that μ belongs to $\sigma(p(T))$. Factor $p(\lambda) - \mu$ as $c \cdot \prod_{k=1}^{n}(\lambda - y_k)$ to obtain the following composition of operators:

$$p(T) - \mu \cdot \mathrm{I} = c[(T - y_1 \cdot \mathrm{I}) \circ \cdots \circ (T - y_n \cdot \mathrm{I})].$$

Since $p(T) - \mu \cdot \mathrm{I}$ is noninvertible, there is at least one k for which $T - y_k \cdot \mathrm{I}$ is noninvertible. Therefore y_k belongs to $\sigma(T)$ and, since $\mu = p(y_k)$, μ belongs to $p(\sigma(T))$.

□

Definition. *A bounded symmetric operator T is said to be* **positive definite** *provided its lower spectral bound, $m(T)$, is positive, and said to be nonnegative provided $m(T) \geq 0$.*

Proposition 6. *Let T in $\mathcal{L}(H)$ be symmetric and positive definite. Then T is invertible.*

Proof. Let m and M be the spectral bounds of T. By definition,

$$\langle Th, h\rangle \geq m(T)\langle h, h\rangle \text{ for all } h \text{ in } H.$$

Therefore, by the Cauchy-Schwarz Inequality,

$$\|Tu - Tv\| \geq m(T)\|u - v\| \text{ for all } u, v \text{ in } H. \tag{19}$$

Hence, since $m(T) > 0$, T is one-to-one and has a closed range. But, since T is symmetric, $[T(H)]^{\perp} = \ker T$. Since $T(H)$ is closed and its orthogonal complement is $\{0\}$, $T(H) = H$. Thus T is one-to-one and onto. □

To a bounded symmetric operator T on H there is associated a real quadratic form $Q = Q(T)$ on H defined by $Q(h) \equiv \langle Th, h\rangle$ for all h in H. The norm of Q, which is denoted by $\|Q\|$, is defined by

$$\|Q\| \equiv \sup_{h \in H, \|h\|=1} |Q(h)|.$$

Lemma 7. *Let T in $\mathcal{L}(H)$ be nonnegative and symmetric and Q be its associated quadratic form. Then*

$$\|Th\|^2 \leq \|Q\|\langle Th, h\rangle \textit{ for all } h \textit{ in } H. \tag{20}$$

Proof. There is the following generalization of the Cauchy-Schwarz Inequality:

$$|\langle Tu,\, v\rangle|^2 \leq \langle Tu,\, u\rangle\langle Tv,\, v\rangle \text{ for all } u, v \text{ in } H. \tag{21}$$

To verify this, for u, v in H, observe that, since $T \geq 0$, $q(t) \equiv \langle T(u+tv),\, u+tv\rangle \geq 0$ for all real numbers t. Therefore the discriminant of the polynomial $q(t) = \langle Tu,\, u\rangle + 2tRe\langle Tu,\, v\rangle + t^2\langle Tv,\, v\rangle$ is nonpositive, that is, $[Re\langle Tu,\, v\rangle]^2 \leq \langle Tu,\, u\rangle\langle Tv,\, v\rangle$. In this inequality, substitute λu for u, where the complex number λ is chosen so the $|\lambda| = 1$ and $\lambda\langle Tu,\, v\rangle = |\langle Tu,\, v\rangle|$, to obtain (21).

For h in H, substitute h for u and Th for v in (21), note that $|\langle T(Th),\, Th\rangle| \leq \|Q\|\|Th\|^2$, and so deduce that $\|Th\|^4 \leq \langle Th,\, h\rangle\|Q\|\|Th\|^2$. Therefore (20) holds. □

Theorem 8 (The Spectral Boundary Theorem)**.** *Let T in $\mathcal{L}(H)$ be symmetric and have spectral bounds m and M. Then the spectrum of T, $\sigma(T)$, is a closed subset of the interval $[m,\, M]$ that contains its end-points m and M.*

Proof. To show that $\sigma(T)$ is closed, we show that its complement, its resolvent $\rho(T)$, is open. To do so, first observe that if S belongs to $\mathcal{L}(H)$ and $\|S\| < 1$, then $\mathrm{I} - S$ is invertible. Indeed, by the completeness of $\mathcal{L}(H)$, the Neumann series $\sum_{k=0}^{\infty} S^k$ converges. We compose to verify that this series is the inverse of $\mathrm{I} - S$. Now let λ_0 belong to $\rho(T)$. We claim that $T - \lambda \cdot \mathrm{I}$ is invertible if $|\lambda - \lambda_0|\|(T - \lambda_0 \cdot \mathrm{I})^{-1}\| < 1$, and hence λ_0 belongs to the interior of $\rho(T)$. The verification of this claim follows from from the preceding observation by observing if we define $S = (\lambda - \lambda_0)(T - \lambda_0 \cdot \mathrm{I})^{-1}$, then $\|S\| < 1$ and

$$T - \lambda \cdot \mathrm{I} = T - \lambda_0 \cdot \mathrm{I} - (\lambda - \lambda_0) \cdot \mathrm{I} = (T - \lambda_0 \cdot \mathrm{I})(\mathrm{I} - S).$$

To show that $\sigma(T)$ is real, let $\lambda = \alpha + \imath\beta$, with α and β real and $\beta \neq 0$. We claim that $T - \lambda \cdot \mathrm{I}$ is invertible. Indeed, let h belong to H. Since

$$\langle Th - \lambda h,\, h\rangle = \langle Th,\, h\rangle - \alpha\langle h,\, h\rangle - \imath\beta\langle h,\, h\rangle,$$

and, by the symmetry of T, $\langle Th,\, h\rangle$ is real, while, by choice, α and β are real, we deduce from the Cauchy-Schwarz Inequality that

$$\|Th - \lambda h\| \geq |\beta|\|h\| \text{ for all } h \text{ in } H. \tag{22}$$

Therefore, since $|\beta| > 0$, $T - \lambda \cdot \mathrm{I}$ is one-to-one and has closed range. This also holds if we replace λ by its complex conjugate, $\overline{\lambda}$. In particular, $\ker(T - \overline{\lambda} \cdot \mathrm{I}) = \{0\}$. But, since T is symmetric,

$$[(T - \lambda \cdot \mathrm{I})H]^{\perp} = \ker(T - \overline{\lambda} \cdot \mathrm{I}) = \{0\}.$$

Therefore, since $(T - \lambda \cdot \mathrm{I})H$ is closed, $T - \lambda \cdot \mathrm{I}$ is one-to-one and onto. We deduce from (22) that its set inverse is bounded, so $T - \lambda \cdot \mathrm{I}$ is invertible. Thus the spectrum of T is real.

To verify the inclusion $\sigma(T) \subseteq [m,\, M]$, first consider $\lambda_0 > M$. Then $\lambda_0 \cdot \mathrm{I} - T$ is positive definite, and hence, by the preceding proposition, λ_0 belongs to the resolvent of T. On the other hand, if $\lambda_0 < m$, then $T - \lambda_0 \cdot \mathrm{I}$ is positive definite, and hence, again by the preceding proposition, λ_0 belongs to the resolvent of T.

To show that M belongs to $\sigma(T)$, observe that since $M \cdot \mathrm{I} - T$ is a nonnegative symmetric operator, according to (20),

$$\|(M \cdot \mathrm{I} - T)h\|^2 \leq Q(M \cdot \mathrm{I} - T) \cdot \langle (M \cdot \mathrm{I} - T)h,\, h\rangle \text{ for all } h \text{ in } H.$$

By the definition of M, there is a sequence $\{h_n\}$ of unit vectors such that $\{\langle (M \cdot \mathrm{I} - T)h_n,\, h_n\rangle\} \to 0$. The above inequality tells us that $\{(M \cdot \mathrm{I} - T)h_n\} \to 0$. Therefore $M \cdot \mathrm{I} - T$ cannot possess an inverse, since an inverse would be continuous. Hence M belongs to $\sigma(T)$. Replacing $M \cdot \mathrm{I} - T$ by $T - m \cdot \mathrm{I}$, the same argument shows that m also belongs to $\sigma(T)$.

□

Proposition 9. *Let T in $\mathcal{L}(H)$ be symmetric and $Q(T)$ be its associated quadratic form. Then $\|T\| = \|Q(T)\|$. In particular, for $c \geq 0$,*

$$\text{if } -c \cdot \mathrm{I} \leq T \leq c \cdot \mathrm{I}, \text{ then } \|T\| \leq c. \tag{23}$$

Proof. For notational simplicity set $\eta = \|Q(T)\|$. If $\eta = 0$, we deduce from (20) that $T = 0$. So consider the case $\eta > 0$. Let h be a unit vector in H. Observe that, by the Cauchy-Schwarz Inequality, $|\langle T(h), h\rangle| \leq \|T(h)\|\|h\| \leq \|T\|$. Thus $\eta \leq \|T\|$. Moreover, by (20), $\|Th\|^2 \leq \eta\|Th\|$, and hence $\|Th\| \leq \eta$. Therefore, $\|T\| \leq \eta$. □

For a bounded linear operator on T on H, its spectral radius, $r_\sigma(T)$, is defined by

$$r_\sigma(T) \equiv \sup\{|\lambda| \mid \lambda \text{ in } \sigma(T)\}.$$

Theorem 10 (The Spectral Radius Theorem)**.** *Let T in $\mathcal{L}(H)$ be symmetric. Then*

$$r_\sigma(T) = \|T\|. \tag{24}$$

Proof. Let m and M be the spectral bounds of T. According to the Spectral Boundary Theorem, $\sigma(T) \subseteq [m, M]$ and both m and M belong to $\sigma(T)$. Therefore $r_\sigma(T) = \max\{|m|, |M|\}$. On the other hand, by the proceeding proposition,

$$\|T\| = \sup\{|\langle Th, h\rangle| \mid \|h\| = 1\} = \max\{|m|, |M|\}.$$

Therefore (24) holds.

□

1.4. **The Family of Spectral Measures for T.**

Lemma 11 (The Spectral Measure Lemma)**.** *Let T in $\mathcal{L}(H)$ be symmetric and the vector h belong to H. Then there is one and only one real finite measure μ_h on $\mathcal{B}(\sigma(T))$, the Borel σ-algebra of the spectrum of T, $\sigma(T)$, such that, for each polynomial p with real coefficients,*

$$\langle p(T)h, h\rangle = \int_{\sigma(T)} p(\lambda)\, d\mu_h(\lambda) \quad \text{and} \quad \|p(T)h\|^2 = \int_{\sigma(T)} p^2(\lambda)\, d\mu_h(\lambda). \tag{25}$$

Proof. For p a polynomial with real coefficients, define

$$\psi(p) \equiv \langle p(T)h, h\rangle.$$

Observe that $p(T)$ is symmetric since p has real coefficients and T is symmetric, and, as a consequence, ψ is real-valued. The functional $p \mapsto \psi(p)$ is linear. There is the following estimate of $|\psi(p)|$:

$$\begin{aligned}
&\text{(by the Cauchy-Schwarz Inequality)} && |\psi(p)| \leq \|p(T)\|\|h\|^2 \\
&\text{(by the Spectral Radius Theorem)} && = r_\sigma(p(T))\|h\|^2 \\
&\text{(by the Spectral Mapping Theorem)} && = \sup\{|p(\lambda)| \mid \lambda \text{ in } \sigma(T)\}\,\|h\|^2 \\
& && = \|p\|_{C(\sigma(T),\mathbf{R})} \cdot \|h\|^2.
\end{aligned}$$

This estimate tells us that if we equip the linear space $C(\sigma(T), \mathbf{R})$ with the maximum norm and let $\mathcal{P}$ be the subspace of restrictions to $\sigma(T)$ of polynomials with real coefficients, then the linear functional $\psi\colon \mathcal{P} \to \mathbf{R}$ is continuous. According to the Spectral Boundary Theorem, $\sigma(T)$ is compact, and so we may appeal to the Weierstrass Approximation Theorem to deduce that $\mathcal{P}$ is a dense subspace of $C(\sigma(T), \mathbf{R})$. We may therefore uniquely extend ψ to a continuous linear functional $\psi\colon C(\sigma(T), \mathbf{R}) \to \mathbf{R}$. We claim that this functional is positive, in the sense that if $f \geq 0$ on $\sigma(T)$, then $\psi(f) \geq 0$. Indeed, first let $f = p$ be a polynomial with real coefficients that is nonnegative on $\sigma(T)$. Then, by the Spectral Mapping Theorem, the spectrum of $p(T)$ is nonnegative and therefore, by the Spectral Boundary Theorem, its lower spectral bound is nonnegative, that is, $\langle p(T)h,\, h\rangle \geq 0$. So the operator $p(T)$ is positive. From this we deduce, by the continuity of ψ with respect to the maximum norm and the Weierstrass Approximation Theorem, that the functional ψ is positive.

According to the Riesz-Markov Representation Theorem[1], since ψ is a positive, bounded, linear functional, there is one and only one finite real Borel measure μ_h on the Borel σ-algebra $\mathcal{B}(\sigma(T))$ such that

$$\psi(f) = \int_{\sigma(T)} f(\lambda)\, d\mu_h(\lambda) \text{ for all } f \text{ in } C(\sigma(T), \mathbf{R}).$$

This establishes the left-hand equality of (25), from which we deduce that, for a polynomial p that has real coefficients,

$$\|p(T)h\|^2 = \langle p(T)h,\, p(T)h\rangle$$

(since $p(T)$ is symmetric)
$$= \langle p^2(T)h,\, h\rangle$$

(by the choice of μ_h)
$$= \int_{\sigma(T)} p^2(\lambda)\, d\mu_h(\lambda).$$

Hence the right-hand equality of (25) also holds.

□

We refer to the collection of finite Borel measures $\{\mu_h\}_{h\in H}$ for which (25) holds for each h in H as the family of spectral measures for T.

1.5. **The Functional Calculus: Linearity, Monotonicity and the Product Formula.**

In this section, the only sigma algebra we consider is the collection of Borel subsets of $\mathcal{B}(\sigma(T))$. So we use $L^2(\sigma(T),\, \nu)$ to denote the space $L^2(\sigma(T), \mathcal{B}(\sigma(T)),\, \nu)$.

Lemma 12. *Let T in $\mathcal{L}(H)$ be symmetric, $\nu\colon \mathcal{B}(\sigma(T)) \to [0,\, \infty)$ be a finite, real Borel measure, and $f\colon \sigma(T) \to \mathbf{R}$ be a bounded Borel function. There is a sequence $\{p_n\}$ of polynomials with real coefficients that converges to f in $L^2(\sigma(T),\; \nu)$.*

Proof. The spectral boundary theorem tells us that $\sigma(T)$ is compact. Since $\sigma(T)$ is a compact metric space, by Lusin's Theorem the continuous functions are dense in $L^2(\sigma(T),\; \nu)$. Since f is bounded and ν is finite, f belongs to $L^2(\sigma(T),\; \nu)$. Therefore, there is a sequence of continuous

[1] see, for instance, [11, p. 458]

functions on $\sigma(T)$ that converges in $L^2(\sigma(T),\ \nu)$ to f. We appeal to the Weierstrass Approximation Theorem to obtain a sequence of polynomials with real coefficients that converges to f in $L^2(\sigma(T),\ \nu)$. $\square$

Lemma 13. *Let T in $\mathcal{L}(H)$ be symmetric and $f\colon \sigma(T) \to \mathbf{R}$ be a bounded Borel function. There is a unique bounded symmetric operator $f(T)$ on H such that, for any vector h in H and sequence $\{p_n\}$ of polynomials with real coefficients,*

$$f(T)h = \lim_{n\to\infty} p_n(T)h \quad \textit{if} \quad \{p_n\} \to f \ \textit{in}\ L^2(\sigma(T),\, \mu_h). \tag{26}$$

Proof. Let h belong to H. Let $\{p_n\}$ be a sequence of polynomials with real coefficients that converges to f in $L^2(\sigma(T),\, \mu_h)$. The preceding lemma tells us that there is such a sequence. This sequence is Cauchy in $L^2(\sigma(T),\, \mu_h)$. But, by the right-hand equality in (25), for any m and n,

$$\|p_n(T)h - p_m(T)h\|^2 = \int_{\sigma(T)} |p_n(\lambda) - p_m(\lambda)|^2 d\mu_h(\lambda).$$

Hence $\{p_n(T)h\}$ is Cauchy in H, so that, since H is complete, $\{p_n(T)h\}$ converges to a vector that we denote by w. We deduce from the right-hand equality in (25) that if $\{q_n\}$ another sequence of polynomials that converges to f in $L^2(\sigma(T),\, \mu_h)$, then $\{q_n(T)h\}$ converges to the same vector w. Define $f(T)h \equiv w$. Therefore, by definition, (26) holds. It remains to show that the correspondence $h \mapsto f(T)h$ defines a bounded linear symmetric operator on H.

To show that $f(T)$ is linear, choose u, v in H and α, β in $\mathbf{R}$. Define a finite measure ν on $\mathcal{B}(\sigma(T))$ by

$$\nu = \mu_u + \mu_v + \mu_{\alpha u+\beta v},$$

where $\{\mu_h\}_{h\in H}$ is the family of spectral measures associated with T. The preceding lemma tells us that there is a sequence $\{p_n\}$ of polynomials with real coefficients that converges to f in $L^2(\sigma(T),\, \nu)$. Then $\{p_n\}$ converges to f in $L^2(\sigma(T),\, \mu_u)$, in $L^2(\sigma(T),\, \mu_v)$, and in $L^2(\sigma(T),\, \mu_{\alpha u+\beta v})$. Now, for each n,

$$p_n(T)[\alpha u + \beta v] = \alpha p_n(T)u + \beta p_n(T)v.$$

The left-hand side converges to $f(T)[\alpha u + \beta v]$ since $\{p_n\}$ converges to f in $L^2(\sigma(T),\, \mu_{\alpha u+\beta v})$, while the right-hand side converges to $\alpha f(T)u + \beta f(T)v$ since $\{p_n\}$ converges to f in both $L^2(\sigma(T),\, \mu_u)$ and $L^2(\sigma(T),\, \mu_v)$. Therefore $f(T)$ is linear.

To verify symmetry, observe since each $p_n(T)$ is symmetric, for each n, $\langle p_n(T)u,\, v\rangle = \langle u,\, p_n(T)v\rangle$. The sequence $\{p_n\}$ converges to f in both $L^2(\sigma(T),\, \mu_u)$ and $L^2(\sigma(T),\, \mu_v)$. Use the continuity of the inner-product in H to take the limit and deduce that $f(T)$ is symmetric.

It remains to show that $f(T)$ is bounded. To do so, set $p(\lambda) \equiv 1$ in (25), and conclude that $\mu_h(\sigma(T)) = \|h\|^2$. We deduce from (25) and the continuity of the norm in H and in $L^2(\sigma(T),\, \mu_v)$ that

$$\|f(T)h\|^2 = \lim_{n\to\infty} \|p_n(T)h\|^2 = \lim_{n\to\infty} \int_{\sigma(T)} p_n^2(\lambda) d\,\mu_h = \int_{\sigma(T)} f^2(\lambda) d\,\mu_h. \tag{27}$$

Hence $\|f(T)h\| \le c\|h\|$ for all $h \in H$, where $c = \sup\{|f(\lambda|\mid \lambda \text{ in } \sigma(T)\}$. $\square$

Theorem 14. *Let T in $\mathcal{L}(H)$ be symmetric and, for a bounded Borel function $f\colon \sigma(T) \to \mathbf{R}$, let $f(T)$ be the bounded symmetric operator on H defined by (26). Then, for all h in H,*

$$\langle f(T)h,\, h\rangle = \int_{\sigma(T)} f(\lambda)\, d\mu_h(\lambda) \quad \textit{and} \qquad \|f(T)h\|^2 = \int_{\sigma(T)} f^2(\lambda)\, d\mu_h(\lambda), \tag{28}$$

where $\{\mu_h\}_{h\in H}$ is the family of spectral measures for T. The transformation $f \mapsto f(T)$ possesses the following properties: for f and g bounded Borel function on $\sigma(T)$ and real numbers α and β,

(i) Linearity:

$$(\alpha f + \beta g)(T) = \alpha f(T) + \beta g(T);$$

(ii) The Product Property:

$$(f \cdot g)(T) = f(T) \circ g(T);$$

(iii) Commutativity: The operator $f(T)$ commutes with T, and indeed commutes with any bounded linear operator on H that commutes with T;

(iv) Monotonicity: If $f \geq 0$ on $\sigma(T)$, then $f(T) \geq 0$.

Proof. Let h belong to H. According to Lemma 12, we may choose sequences of polynomials with real coefficients, $\{p_n\}$ and $\{q_n\}$, that converge in $L^2(\sigma(T), \mu_h)$ to f and g respectively.

Since μ_h is a finite measure, convergence in L^2 implies convergence in L^1. Therefore, since the left-hand equality in (28) holds if $f(T)$ is replaced by $p_n(T)$, passage to the limit establishes it in general. The right-hand equality in (28) was already established as (27).

The linearity property follows from linearity of convergence in H and in L^2.

To verify the commutativity property, let S in $\mathcal{L}(H)$ commute with T. Define a finite measure ν on $\mathcal{B}(\sigma(T))$ by $\nu = \mu_h + \mu_{Sh}$. By Lemma 12, we may choose a sequence of polynomials with real coefficients $\{r_n\}$ that converges in $L(\sigma(T), \nu)$ to f. Now S commutes with $r_n(T)$. Since $\{r_n\}$ converges to f in both $L^2(\sigma(T), \mu_h)$ and in $L^2(\sigma(T), \mu_{Sh})$

$$S(f(T)h) = \lim_{n\to\infty} S(r_n(T)h) = \lim_{n\to\infty} r_n(T)(Sh) = f(T)(Sh).$$

To verify the product property, observe that since T is symmetric and the polynomials p_n and q_n have real coefficients, for all n,

$$\langle q_n(T)h,\, p_n(T)h\rangle = \langle [p_n \cdot q_n](T)h,\, h\rangle. \tag{29}$$

Every sequence that is $L^2(\sigma(T),\ \nu)$ convergent has a subsequence that converge pointwise to the limit function outside of a set of ν-measure 0. Therefore, by possibly choosing subsequenes, we may assume the sequences $\{p_n\}$ and $\{q_n\}$ converge pointwise to f and g respectively, outside a set of μ_h measure 0. Moreover, since f and g are pointwise bounded on $\sigma(T)$, we may choose the sequences $\{p_n\}$ and $\{q_n\}$ to be uniformly pointwise bounded on $\sigma(T)$. Therefore, by the Bounded Convergence Theorem, $\{p_n \cdot q_n\}$ converges in $L^2(\sigma(T), \mu_h)$ to $f \cdot g$. Consequently,

$$\langle g(T)h,\, f(T)h\rangle = \lim_{n\to\infty} \langle q_n(T)h,\, p_n(T)h\rangle = \lim_{n\to\infty} \langle [p_n \cdot q_n](T)h,\, h\rangle = \langle [f \cdot g](T)h,\, h\rangle,$$

so that, since $f(T)$ is symmetric,

$$\langle (f(T) \circ g(T))h,\, h\rangle = \langle g(T)h,\, f(T)h\rangle = \langle [f \cdot g](T)h,\, h\rangle.$$

Since $f(T)$ and $g(T)$ are symmetric and commute, $f(T) \circ g(T)$ is symmetric. The quadratic form induced by the symmetric operator $f(T) \circ g(T) - [f \cdot g](T)$ vanishes. We deduce from (23) that $f(T) \circ g(T) = [f \cdot g](T)$.

To verify the monotonicity property, let $f\colon \sigma(T) \to \mathbf{R}$ be a nonnegative, bounded Borel function. Then so is $\sqrt{f}\colon \sigma(T) \to \mathbf{R}$. By the product property and the symmetry $\sqrt{f}(T)$,

$$\langle f(T)h, h\rangle = \langle [\sqrt{f} \cdot \sqrt{f}](T)h, h\rangle = \langle \sqrt{f}(T) \circ \sqrt{f}(T)h, h\rangle = \langle \sqrt{f}(T)h, \sqrt{f}(T)h\rangle \geq 0,$$

for all h in H; that is, $f(T) \geq 0$.

$\square$

Let T in $\mathcal{L}(H)$ be symmetric and $f\colon \mathbf{R} \to \mathbf{R}$ a bounded Borel function. We define $f(T)$ to be $\hat{f}(T)$, where $\hat{f}$ is the restriction of f to $\sigma(T)$. In this way, the spectral calculus, possessing all of the above properties, is extended to bounded Borel functions $f\colon \mathbf{R} \to \mathbf{R}$.

1.6. Existence and Uniqueness of a Spectral Resolution.

Definition. *Let T in $\mathcal{L}(H)$ be symmetric. A right-continuous, increasing path of orthogonal projections on H, $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$, with the property that $h = \int_{-\infty}^{\infty} d\mathcal{E}h$ for all h in H, is called a* **spectral resolution** *of T provided*

$$Th = \int_{-\infty}^{\infty} \lambda \cdot d\mathcal{E}_\lambda h \text{ for all } h \text{ in } H. \tag{30}$$

In this section, we prove that each symmetric operator in $\mathcal{L}(H)$ has one and only one spectral resolution.

For a Borel subset E of $\mathbf{R}$, χ_E is defined to be the characteristic function of E, which takes the value 1 on E and 0 on the complement in $\mathbf{R}$ of E. Observe that if I is an interval, then χ_I is a bounded Borel function on $\mathbf{R}$ and so, if T in $\mathcal{L}(H)$ is symmetric, the bounded symmetric operator $\chi_I(T)$ is defined.

Proposition 15. *Let T in $\mathcal{L}(H)$ be symmetric. For each real number λ, define*

$$\mathcal{E}_\lambda \equiv \chi_{(-\infty,\,\lambda]}(T). \tag{31}$$

Then $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ is right-continuous, increasing path of orthogonal projections on H with the property that $h = \int_{-\infty}^{\infty} d\mathcal{E}h$ for all h in H. Moreover, T maps each $\mathcal{E}_\lambda(H)$ into itself and

$$\|Th - \lambda_0 h\| \le (\beta-\alpha)\|h\| \text{ for all } \alpha<\beta, h \text{ in } \mathcal{E}_\beta(H)\cap[\mathcal{E}_\alpha(H)]^\perp \text{ and } \lambda_0 \text{ in } [\alpha,\beta].$$

Proof. Observe that, for α, β in $\mathbf{R}$,

$$\chi_{(-\infty,\,\alpha]} \cdot \chi_{(-\infty,\,\beta]} = \chi_{(-\infty,\,\min\{\alpha,\beta\}]}.$$

Therefore, by the product property of the functional calculus, each $\mathcal{E}_\lambda$ is a projection, and is an orthogonal projection since it is symmetric. We deduce from the monotonicity property of the functional calculus that the path $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ is increasing.

To verify right-continuity, for λ_0 in $\mathbf{R}$, let the sequence $\{\lambda_n\}$ in (λ_0, ∞) converge to λ_0. Let h belong to H. According to conclusion (28) of Theorem 14, for each n,

$$\|(\mathcal{E}_{\lambda_n} - \mathcal{E}_{\lambda_0})h\|^2 = \int_{\sigma(T)} \chi^2_{(\lambda_0,\lambda_n]} d\mu_h = \mu_h(\sigma(T)\cap(\lambda_0,\lambda_n]).$$

Therefore, by the continuity property of the finite Borel measure μ_h, $\lim_{n\to\infty} \mathcal{E}_{\lambda_n}h = \mathcal{E}_{\lambda_0}h$.

Let m and M be the spectral bounds of T. For $\beta \ge M$, $\chi_{(-\infty,\beta]}$ takes the value 1 on $\sigma(T)$ and therefore $\mathcal{E}_\beta = \mathrm{I}$. For $\alpha < m$, $\chi_{(-\infty,\alpha]}$ takes the value 0 on $\sigma(T)$ and therefore $\mathcal{E}_\alpha = 0$. Therefore, for all h in H, $\alpha < m$ and $\beta \ge M$, $\int_\alpha^\beta d\mathcal{E}_\lambda h = h$ and so $\int_{-\infty}^{\infty} d\mathcal{E}_\lambda h = h$.

By the commutativity properties of the functional calculus, T commutes with each $\mathcal{E}_\lambda$ and hence maps each $\mathcal{E}_\lambda(H)$ into itself. It remains to verify the final assertion. Let $\alpha < \beta$. Observe that, if we define $g(\lambda) \equiv \lambda \cdot \chi_{(\alpha,\,\beta]}(\lambda)$, then

$$\alpha \cdot \chi_{(\alpha,\,\beta]}(\lambda) \le g(\lambda) \le \beta \cdot \chi_{(\alpha,\,\beta]}(\lambda) \text{ for all } \lambda.$$

By the linearity property of the functional calculus, $\chi_{(\alpha,\,\beta]}(T) = \mathcal{E}_\beta - \mathcal{E}_\alpha$, and by the product product of the functional calculus, $g(T) = T\circ(\mathcal{E}_\beta - \mathcal{E}_\alpha)$. Therefore, by the monotonicity property of the functional calculus,

$$\alpha\cdot(\mathcal{E}_\beta - \mathcal{E}_\alpha) \le T\circ(\mathcal{E}_\beta - \mathcal{E}_\alpha) \le \beta\cdot(\mathcal{E}_\beta - \mathcal{E}_\alpha),$$

that is,

$$\alpha \cdot \mathrm{I} \leq T \leq \beta \cdot \mathrm{I} \text{ on } (\mathcal{E}_\beta - \mathcal{E}_\alpha)H.$$

Therefore, for $\alpha \leq \lambda_0 \leq \beta$,

$$-(\beta-\alpha)\cdot \mathrm{I} \leq (\alpha-\lambda_0)\cdot \mathrm{I} \leq T - \lambda_0 \cdot \mathrm{I} \leq (\beta-\lambda_0)\cdot \mathrm{I} \leq (\beta-\alpha)\cdot \mathrm{I} \text{ on } (\mathcal{E}_\beta - \mathcal{E}_\alpha)H.$$

Since, by (9), $(\mathcal{E}_\beta - \mathcal{E}_\alpha)H = \mathcal{E}_\beta(H) \cap [\mathcal{E}_\alpha(H)]^\perp$, an appeal to (23) verifes the final assertion. □

Lemma 16. *Let $\{\mathcal{E}_\lambda\}_{\lambda\in[a,\,b]}$ be an increasing path of orthogonal projections on H for which $\mathcal{E}_a = 0$ and $\mathcal{E}_b = \mathrm{I}$. Define*

$$T = \int_a^b \lambda d\,\mathcal{E}_\lambda. \tag{32}$$

Then, for every polynomial p with real coefficients,

$$p(T) = \int_a^b p(\lambda)\, d\mathcal{E}_\lambda. \tag{33}$$

Proof. Define $f(\lambda) \equiv \lambda$. Proposition 2 tells us that the integral in (32) is properly defined. Let π be a partition of $[a,\,b]$ and c be a choice set for π. Let j be a natural number. We deduce from (11) that

$$[\mathrm{Sum}(f,\,\mathcal{E},\,\pi,\,c)]^j = \left[\sum_{k=1}^n c_k \cdot (\mathcal{E}_{\lambda_k} - \mathcal{E}_{\lambda_{k-1}})\right]^j = \sum_{k=1}^n c_k^j \cdot (\mathcal{E}_{\lambda_k} - \mathcal{E}_{\lambda_{k-1}}). \tag{34}$$

Proposition 2 tells us that f and f^j are Stieltjes integrable with respect to $\{\mathcal{E}_\lambda\}_{\lambda\in[a,\,b]}$. Take the limit as $\mathrm{gap}\,\pi \to 0$ to deduce that

$$T^j = \int_a^b \lambda^j \cdot\, d\mathcal{E}_\lambda. \tag{35}$$

Since $\mathcal{E}_a = 0$ and $\mathcal{E}_b = \mathrm{I}$, (35) also holds for $j = 0$. Thus, in view of (35), (33) holds. □

Rather than proving uniqueness of the spectral resolution by pursuing the correspondence between the resolution and the family of spectral measures, we establish uniqueness by simply following the same argument used for the Stieltjes integral of a real valued function.

Lemma 17 (The Uniqueness Lemma)**.** *Let T in $\mathcal{L}(H)$ be symmetric and $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ and $\{\mathcal{E}'_\lambda\}_{\lambda\in\mathbf{R}}$ be spectral resolutions of T. Then $\mathcal{E}_\lambda = \mathcal{E}'_\lambda$ for all λ.*

Proof. Let m and M be the spectral bounds of T, choose $a < m$ and let $b = M$. We deduce from Theorem 4 and Lemma 15 that

$$\mathcal{E}'_a = \mathcal{E}_a = 0 \text{ and } \mathcal{E}'_b = \mathcal{E}_b = \mathrm{I}\,. \tag{36}$$

Therefore the paths $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ and $\{\mathcal{E}'_\lambda\}_{\lambda\in\mathbf{R}}$ are constant on the intervals $(-\infty, a]$ and $[b, \infty)$, and so

$$\int_a^b \lambda \cdot\, d\mathcal{E}'_\lambda h = \int_a^b \lambda \cdot\, d\mathcal{E}_\lambda h \text{ for all } h \text{ in } H.$$

Let h belong to H. According to Lemma 16

$$\int_a^b p(\lambda)\cdot d\langle \mathcal{E}_\lambda h,\, h\rangle = \int_a^b p(\lambda)\cdot d\langle \mathcal{E}'_\lambda,\, h\rangle h \text{ if } p \text{ is a polynomial with real coefficients.}$$

Define $\psi(\lambda) = \langle \mathcal{E}_\lambda h,\, h\rangle - \langle {\mathcal{E}'}_\lambda h,\, h\rangle$ for all λ in $[a,\, b]$. From the above and the Weierstrass Approximation Theorem we deduce that

$$\int_a^b f(\lambda)\, d\psi(\lambda) = 0 \text{ for all } f \text{ in } C([a,\, b], \mathbf{R}). \tag{37}$$

Therefore

$$\psi(a) = \psi(x_0) = \psi(b) \quad \text{if} \quad \psi\colon [a,\, b] \to \mathbf{R} \text{ is continuous at the point } x_0 \text{ in } [a,\, b]. \tag{38}$$

Indeed, to see, say, that $\psi(x_0) = \psi(b)$, let $\epsilon > o$ and let f_ϵ be a continuous, increasing functions from $[a,\, b]$ to $[0, 1]$ that rises from 0 to 1 on $[x_0, x_0 + \epsilon]$. Use (37), with $f = f_\epsilon$, and take the limit as $\epsilon \to 0$ to see that $\psi(x_0) = \psi(b)$. Now, since ψ is the difference of right-continuous functions, it is right-continuous, and since it is the difference of increasing functions, $\psi\colon [a,\, b] \to \mathbf{R}$ is continuous at all but possibly a countable number of points. Hence, by (38), $\psi\colon [a,\, b] \to \mathbf{R}$ is constant. Since

$$\psi(b) = \langle \mathcal{E}_b h,\, h\rangle - \langle {\mathcal{E}'}_b h,\, h\rangle = \|h\|^2 - \|h\|^2 = 0,$$

$\psi \equiv 0$ on $[a,\, b]$. But h was arbitrarily chosen in H. Therefore, for each λ in $[a,\, b]$, the operator $\mathcal{E}_\lambda - \mathcal{E}'$ is symmetric and

$$\langle (\mathcal{E}_\lambda - {\mathcal{E}'}_\lambda)h,\, h\rangle = 0 \text{ for all } h \text{ in } H.$$

Thus, by (23), $\mathcal{E}_\lambda = {\mathcal{E}'}_\lambda$.

□

Theorem 18 (The Spectral Resolution of a Bounded Symmetric Operator)**.** *Let H be a complex Hilbert space and T in $\mathcal{L}(H)$ be symmetric. Then T has one and only one spectral resolution.*

Proof. For each real number λ, define $\mathcal{E}_\lambda \equiv \chi_{(-\infty,\, \lambda]}(T)$. We deduce from Proposition 15 together with Theorem 4 that $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ is a spectral resolution of T. The Uniqueness Lemma asserts that the spectral resolution is unique.

□

2. The Spectral Resolution of a Self-Adjoint Operator

2.1. The Spectral Resolution and Estimates of $Th - \lambda h$.

Definition. *For D a linear subspace of H, a linear operator $T\colon D \to H$ is said to be* **symmetric** *provided*

$$\langle Tu,\, v\rangle = \langle u,\, Tv\rangle \textit{ for all } u, v \textit{ in } D.$$

If $T\colon D \to H$ is symmetric and D is dense in H, the adjoint operator $T^*\colon D(T^*) \to H$ is defined as follows: Let $D(T^*)$ to be the collection of vectors f in H for which there is a vector g in H such that

$$\langle Tu,\, f\rangle = \langle u,\, g\rangle \text{ for all } u \text{ in } D.$$

Since D is dense in H, there can be only one such vector g. We define $T^*(f) = g$.

Definition. *A densely defined symmetric operator $T\colon D \to H$ is said to be* **self-adjoint** *provided $T = T^*$. It is easy to see that the self-adjointness of T is equivalent to the assertion that T has no proper symmetric extension.*

Definition. *Let $T\colon D \subseteq H \to H$ be self-adjoint. Let $\{H_n\}_{n=1}^{\infty}$ be a sequence of closed subspaces of H such that for each n, $H_n \subseteq H_{n+1}$, $H_n \subseteq D$, T maps H_n into itself and, for each h in H, $\lim_{n\to\infty} Q_n(h) = h$, where each Q_n is the orthogonal projection of H onto H_n. We call the sequence of operators $\{T\colon H_n \to H_n\}_{n=1}^{\infty}$ a* **sequence of bounded approximations** *of $T\colon D \subseteq H \to H$.*

Regarding the above definition, a theorem of Hellinger and Toeplitz, whose proof is an immediate consequence of the Closed Graph Theorem (see [4]) tells us that since each H_n is a Hilbert space and each approximation $T\colon H_n \to H_n$ is symmetric, each approximation is bounded. This, together with the following theorem, justifies that name "bounded approximation."

The proof of the existence of a spectral resolution for a self-adjoint operator is based on the following theorem, whose proof we postpone until the final section.

Theorem 19 (The Bounded Approximation Theorem)**.** *Let $T\colon D \subseteq H \to H$ be self-adjoint and $\{T\colon H_n \to H_n\}_{n=1}^{\infty}$ be a sequence of bounded approximations of $T\colon D \subseteq H \to H$. Then the domain D comprises those h in H for which $\{TQ_nh\}_{n=1}^{\infty}$ is bounded and, for each such h, $Th = \lim_{n\to\infty} TQ_nh$. Furthermore, every self-adjoint operator possesses a sequence of bounded approximations.*

Definition. *Let the operator $T\colon D \subseteq H \to H$ be self-adjoint. A right-continuous, increasing path of orthogonal projections on H $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$, with the property that $h = \int_{-\infty}^{\infty} d\,\mathcal{E}_\lambda h$ for all h in H, is called a* **spectral resolution** *of T provided*

$$D = \left\{ h \textit{ in } H \;\middle|\; \int_{-\infty}^{\infty} \lambda^2 d\langle \mathcal{E}_\lambda h,\, h\rangle < \infty \right\} \textit{ and } Th = \int_{-\infty}^{\infty} \lambda\, d\mathcal{E}_\lambda h \textit{ for all } h \textit{ in } D.$$

The above definition of spectral resolution is precisely that of von Neumann ([6, p. 119]). There is the following direct extension of Theorem 4.

Theorem 20. *Let $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ be a right-continuous, increasing path of orthogonal projections on H with the property that $h = \int_{-\infty}^{\infty} d\mathcal{E}_\lambda h$ for all h in H. Let the operator $T\colon D \subseteq H \to H$ be self-adjoint. Then*

$$D = \left\{ h \text{ in } H \;\middle|\; \int_{-\infty}^{\infty} \lambda^2 \, d\langle \mathcal{E}_\lambda h,\, h\rangle < \infty \right\} \quad \text{and} \quad Th = \int_{-\infty}^{\infty} \lambda \, d\mathcal{E}_\lambda h \text{ for all } h \text{ in } D \tag{39}$$

if and only if, for $\alpha < \beta$, T maps $\mathcal{E}_\beta(H) \cap [\mathcal{E}_\alpha(H)]^\perp$ into itself and

$$\|Th - \lambda_0 h\| \le (\beta - \alpha)\|h\| \text{ for all } h \text{ in } \mathcal{E}_\beta(H) \cap [\mathcal{E}_\alpha(H)]^\perp \text{ and } \lambda_0 \text{ in } [\alpha, \beta]. \tag{40}$$

Proof.

Claim 1: If either (39) or (40) is satisfied then, for $\alpha < \beta$, the subspace $[\mathcal{E}_\beta - \mathcal{E}_\alpha](H)$ is contained in D and mapped by T into itself. Of course, if (40) is satisfied, this claim is true by assumption. Assume (39) holds. Observe that if $\alpha < \beta$ and h belongs to $[\mathcal{E}_\beta - \mathcal{E}_\alpha](H)$, then, by (14)

$$\int_{-\infty}^{\infty} \lambda^2 \, d\langle \mathcal{E}_\lambda h,\, h\rangle = \int_{\alpha}^{\beta} \lambda^2 \, d\langle \mathcal{E}_\lambda h,\, h\rangle \text{ and } \int_{-\infty}^{\infty} \lambda \, d\mathcal{E}_\lambda h = \int_{\alpha}^{\beta} \lambda \, d\mathcal{E}_\lambda h. \tag{41}$$

The above left-hand equality and the left-hand assertion of assumption (39) tell us that, for $\alpha < \beta$, $[\mathcal{E}_\beta - \mathcal{E}_\alpha](H) \subseteq D$, and therefore, by the above right-hand equality and right-hand assertion of assumption (39),

$$Th = \int_{\alpha}^{\beta} \lambda \, d\mathcal{E}_\lambda h \text{ for all } h \text{ in } [\mathcal{E}_\beta - \mathcal{E}_\alpha](H).$$

From this integral representation and the commutativity of the projections, we deduce that $T = T \circ [\mathcal{E}_\beta - \mathcal{E}_\alpha] = [\mathcal{E}_\beta - \mathcal{E}_\alpha] \circ T$ on $[\mathcal{E}_\beta - \mathcal{E}_\alpha](H)$, so that T maps each subspace $[\mathcal{E}_\beta - \mathcal{E}_\alpha](H)$ into itself. This first claim is verified.

Claim 2: For each n, define $Q_n \equiv \mathcal{E}_n - \mathcal{E}_{-n}$ and $H_n \equiv Q_n(H)$. If either (39) or (40) is satisfied, then the sequence of operators $\{T\colon H_n \to H_n\}_{n=1}^{\infty}$ is a sequence of bounded approximations of $T\colon D \subseteq H \to H$. Indeed, since $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ increasing, for each n, $H_n \subseteq H_{n+1}$, and the preceding claim tells us that $H_n \subseteq D$ and T maps H_n into itself. Each H_n is closed since the projections are continuous. Moreover, the assumption that $h = \int_{-\infty}^{\infty} d\mathcal{E}_\lambda h$ for all h in H is equivalent to the assertion $\lim_{n\to\infty} Q_n(h) = h$ for all h in H. This second claim is justified.

Claim 3: For each n, let $\{\mathcal{E}^n_\lambda\}_{\lambda\in\mathbf{R}}$ be the path of restrictions of the path $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ to H_n. Then each $\{\mathcal{E}^n_\lambda\}_{\lambda\in\mathbf{R}}$ is a right-continuous, increasing path of orthogonal projections on H_n with the property that $h = \int_{-\infty}^{\infty} d\mathcal{E}^n_\lambda h$ for all h in H_n. This follows immediately from the assumptions on $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ and the commutativity of the projections which implies that each $\mathcal{E}_\lambda$ maps each H_n into itself.

Fix n. We appeal to Theorem 4, the version of this theorem for bounded symmetric operators, with $T\colon H_n \to H_n$ substituted for $T\colon H \to H$ and $\{\mathcal{E}^n_\lambda\}_{\lambda\in\mathbf{R}}$ substituted for $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$, and also to (41), to conclude that

$$Th = \int_{-\infty}^{\infty} \lambda \, d\mathcal{E}^n_\lambda h = \int_{-n}^{n} \lambda \, d\mathcal{E}_\lambda h \text{ for all } h \text{ in } H_n \tag{42}$$

if and only if

$$\|Th - \lambda_0 h\| \leq (\beta - \alpha)\|h\| \text{ for all } \alpha < \beta, h \text{ in } \mathcal{E}^n_\beta(H_n) \cap [\mathcal{E}^n_\alpha(H_n)]^\perp \text{ and } \lambda_0 \text{ in } [\alpha, \beta]. \tag{43}$$

According to (14), $\int_{-n}^{n} \lambda \, d\mathcal{E}_\lambda Q_n h = \int_{-n}^{n} \lambda \, d\mathcal{E}_\lambda h$, and, by commutativity of the projections, if h belongs to $\mathcal{E}_\beta(H) \cap [\mathcal{E}_\alpha(H)]^\perp$, then each $Q_n h$ belongs to $\mathcal{E}^n_\beta(H) \cap [\mathcal{E}^n_\alpha(H)]^\perp$. Therefore, by the preceding equivalence:

$$TQ_n h = \int_{-n}^{n} \lambda \, d\mathcal{E}_\lambda h \text{ for all } h \text{ in } H \tag{44}$$

if and only if

$$\|TQ_n h - \lambda_0 Q_n h\| \leq (\beta - \alpha)\|Q_n h\| \text{ for all } \alpha < \beta, h \text{ in } \mathcal{E}_\beta(H) \cap [\mathcal{E}_\alpha(H)]^\perp \text{ and } \lambda_0 \text{ in } [\alpha, \beta]. \tag{45}$$

We now prove the equivalence of (39) and (40). First assume (39) holds. Then, for each n, (42) holds and hence so does (44). The preceding equivalence tells us that (45) holds. However, by the right-hand equality of (39), together with the inclusion of $\mathcal{E}_\beta(H) \cap [\mathcal{E}_\alpha(H)]^\perp$ in D,

$$\lim_{n\to\infty} TQ_n h = Th \text{ for all } h \text{ in } \mathcal{E}_\beta(H) \cap [\mathcal{E}_\alpha(H)]^\perp.$$

By assumption, $\lim_{n\to\infty} Q_n h = h$ for all h in H. Take the limit as $n \to \infty$ in (45) to deduce that (40) holds.

Now assume that (40) holds. Then (43) holds and hence so does (45). By the preceding equivalence, (44) holds. We may therefore appeal to (14) and (13) to deduce that, for all n,

$$\|TQ_n h\|^2 = \int_{-n}^{n} \lambda^2 \, d\langle \mathcal{E}_\lambda h, h\rangle \text{ and } TQ_n h = \int_{-n}^{n} \lambda \, d\mathcal{E}_\lambda h \text{ for all } h \text{ in } H. \tag{46}$$

However, claim 2 asserts that $\{T\colon H_n \to H_n\}_{n=1}^\infty$ is a sequence of bounded approximations of $T\colon D \subseteq H \to H$. We may therefore appeal to Theorem 19 for this particular choice of subspaces, $\{H_n\}$. In view of (46), assertion (39) follows from the conclusion of this theorem. ☐

2.2. **Proof of Existence and Uniqueness.** This definition of spectral resolution is precisely the original definition of von Neumann ([6, p. 118]).

Theorem 21 (The Spectral Resolution of an Unbounded Self-adjoint Operators)**.** *Let H be a complex Hilbert space and the operator $T\colon D \subseteq H \to H$ be self-adjoint. Then T has one and only one spectral resolution.*

Proof. We appeal to Theorem 19 to choose a sequence $\{T\colon H_n \to H_n\}$ of bounded approximations of $T\colon D \subseteq H \to H$, and

$$D = \{h \mid \{TQ_n h\}_{n=1}^\infty \text{ is bounded}\} \text{ and } Th = \lim_{n\to\infty} TQ_n h \text{ for } h \text{ in } D. \tag{47}$$

Fix n. The operator $T\colon H_n \to H_n$ is a bounded symmetric operator on the Hilbert space H_n and hence, by Theorem 18, has a unique spectral resolution $\{\mathcal{E}^n_\lambda\}_{\lambda\in\mathbf{R}}$. We appeal to Theorem 4 to conclude that, if $\alpha < \beta$, then

$$\|Th - \lambda_0 h\| \leq (\beta - \alpha)\|h\| \text{ for all } h \text{ in } (\mathcal{E}^n_\beta - \mathcal{E}^n_\alpha)H_n \text{ and all } \lambda_0 \text{ in } [\alpha, \beta]. \tag{48}$$

Fix λ and $k > n$. The uniqueness of spectral resolutions for bounded symmetric operators tells us that $\mathcal{E}^k_\lambda|_{H_n} = \mathcal{E}^n_\lambda$, so that

$$\mathcal{E}^k_\lambda Q_k h - \mathcal{E}^n_\lambda Q_n h = \mathcal{E}^k_\lambda (Q_k h - Q_n h),$$

and hence

$$\|\mathcal{E}^k_\lambda Q_k h - \mathcal{E}^n_\lambda Q_n h\| \leq \|Q_k h - Q_n h\|. \tag{49}$$

Since $\{Q_n h\}$ converges, the sequence $\{\mathcal{E}^n_\lambda Q_n h\}$ is Cauchy. By the completeness of H we may define $\mathcal{E}_\lambda h = \lim_{n\to\infty} \mathcal{E}^n_\lambda Q_n h$.

Claim 1: $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ is a right-continuous, increasing path of orthogonal projections with the property that $\int_{-\infty}^{\infty} d\mathcal{E}_\lambda h = h$ for all h in H. Indeed, it is clear that, for each n, the operator $\mathcal{E}^n_\lambda Q_n$ in $\mathcal{L}(H)$ is an orthogonal projection. Since each $\{\mathcal{E}^n_\lambda Q_n\}_{\lambda\in\mathbf{R}}$ is a path of orthogonal projections, so is the limit $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$. Taking the limit as $k \to \infty$ in (49) we conclude that, for each h and n,

$$\|\mathcal{E}_\lambda h - \mathcal{E}^n_\lambda Q_n h\| \leq \|h - Q_n h\|. \tag{50}$$

From this it is clear that $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ is right continuous and moreover

$$\lim_{\lambda\to\infty} \mathcal{E}_\lambda h = h \text{ and } \lim_{\lambda\to-\infty} \mathcal{E}_\lambda h = 0,$$

so that $\int_{-\infty}^{\infty} d\mathcal{E}_\lambda h = h$ for all h in H. This claim is verified.

Claim 2: If h belongs to $(\mathcal{E}_\beta - \mathcal{E}_\alpha)H$ than, for all n, $Q_n h$ belongs to $(\mathcal{E}^n_\beta - \mathcal{E}^n_\alpha)H_n$. Indeed, fix n and λ. For each $k > n$, $\mathcal{E}^k_\lambda|_{H_n} = \mathcal{E}^n_\lambda$, and therefore the restriction of Q_n to H_k commutes with $\mathcal{E}^k_\alpha$, so that $Q_n \mathcal{E}^k_\lambda Q_k = \mathcal{E}^n_\lambda Q_n$. Therefore

$$Q_n \mathcal{E}_\lambda = Q_n \lim_{k\to\infty} \mathcal{E}^k_\lambda Q_k = \lim_{k\to\infty} Q_n \mathcal{E}^k_\lambda Q_k = \mathcal{E}^n_\lambda Q_n.$$

This suffices to establish the claim.

Claim 3: For $\alpha < \beta$, the subspace $(\mathcal{E}_\beta - \mathcal{E}_\alpha)H$ is contained in D and mapped by T into itself. Indeed, let h belong to $(\mathcal{E}_\beta - \mathcal{E}_\alpha)H$. Fix n. According to the preceding claim, $Q_n h$ belongs to $(\mathcal{E}^n_\beta - \mathcal{E}^n_\alpha)H_n$. We appeal to (48) to conclude that

$$\|TQ_n h - \lambda_0 Q_n h\| \leq (\beta - \alpha)\|Q_n h\| \text{ for all } \lambda_0 \text{ in } [\alpha, \beta]. \tag{51}$$

Since $\{Q_n h\}$ is bounded, we deduce that $\{TQ_n h\}_{n=1}^{\infty}$ is bounded. We appeal to (47) to conclude that h belongs to D and $Th = \lim_{n\to\infty} TQ_n h$. Since $\{\mathcal{E}^n_\lambda\}$ is a spectral resolution of $T\colon H_n \to H_n$, T maps $(\mathcal{E}^n_\beta - \mathcal{E}^n_\alpha)H_n$ into itself. Since $Q_n h$ belongs to $(\mathcal{E}^n_\beta - \mathcal{E}^n_\alpha)H_n$ so does $TQ_n h$. However, $(\mathcal{E}^n_\beta - \mathcal{E}^n_\alpha)H_n$ is contained in the closed subspace $(\mathcal{E}_\beta - \mathcal{E}_\alpha)H$ and so $Th = \lim_{n\to\infty} TQ_n h$ also belongs to this subspace. This claim is verified.

We appeal to Theorem 4 to conclude that $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ is a spectral resolution of $T\colon D \subseteq H \to H$. It remains to prove uniqueness. Suppose $\{\mathcal{E}'_\lambda\}_{\lambda\in\mathbf{R}}$ also is a spectral resolution of the operator $T\colon D \subseteq H \to H$. Then each $\mathcal{E}'_\lambda$ commutes with T. The spectral resolution $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ was chosen to have the property that if a bounded linear operator S commutes with T, then it also commutes with each $\mathcal{E}_\lambda$. From this we deduce that the restriction of $\{\mathcal{E}'_\lambda\}_{\lambda\in\mathbf{R}}$ to each H_k

is a spectral resolution of the bounded symmetric operator $T\colon H_k \to H_k$. By the uniqueness assertion of Theorem 18, for all k and λ, $\mathcal{E}'_\lambda = \mathcal{E}'_\lambda$ on H_k. Since the union of the H_k's is dense in H, the paths $\{\mathcal{E}'_\lambda\}_{\lambda\in\mathbf{R}}$ and $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ are equal. □

2.3. **Proof of the the Bounded Approximation Theorem.** The results in this section, except von Neuman's theorem, Theorem 23, and strategy of proof, are adaptations of the fundamental results of Riesz and Lorch ([9]). It is convenient to first prove the following property of bounded approximations, and then turn to proving the existence of a bounded approximation for each self-adjoint operator.

Theorem 22. *Let $T\colon D \subseteq H \to H$ be self-adjoint and $\{T\colon H_n \to H_n\}_{n=1}^\infty$ be a sequence of bounded approximations of T. If, for each n, Q_n is the orthogonal projection of H onto H_n, the domain D comprises those h in H for which $\{TQ_nh\}_{n=1}^\infty$ is bounded, and for such h, $Th = \lim_{n\to\infty} TQ_nh$.*

Proof. Let n and k be natural numbers. Since T is symmetric and maps H_{n+k} into itself and H_n into itself, T maps $H_{n+k}\cap H_n^\perp$ into itself. Moreover, according to (5), $Q_{n+k} - Q_n$ is the orthogonal projection of H onto $H_{n+k}\cap H_n^\perp$. Thus, for each h,

$$\|(TQ_{n+k})h - (TQ_n)h\|^2 = \|((TQ_{n+k})h\|^2 - \|(TQ_n)h\|^2.$$

Hence, since H is complete, the sequence $\{TQ_nh\}_{n=1}^\infty$ is bounded if and only if it converges. Therefore, if we define $D' \equiv \{h \text{ in } H \mid \{TQ_nh\}_{n=1}^\infty \text{ is bounded}\}$, we may define $T'\colon D' \subseteq H \to H$ by setting $T'(h) = \lim_{n\to\infty} TQ_nh$ for each h in D'. Clearly, for each n, $H_n \subseteq D'$, so that D' is dense in H and, since T is symmetric, so is T'.

Claim 1: The operator $T'\colon D' \subseteq H \to H$ is self-adjoint. Indeed, let f and g in H have the property that

$$\langle T'u, f\rangle = \langle u, g\rangle \text{ for all } u \text{ in } D'. \tag{52}$$

Since $T = T'$ on $Q_n(H)$, and T is symmetric and maps $Q_n(H)$ into $Q_n(H)$, TQ_nf belongs to $H_n \subseteq D'$, so that

$$\|TQ_nf\|^2 = \langle TQ_nf, TQ_nf\rangle = \langle T(TQ_n)f), f\rangle = \langle TQ_nf, g\rangle \text{ for all } n.$$

Therefore, by the Cauchy-Schwarz inequality, $\|TQ_nf\| \le \|g\|$, for all n. So f belongs to D'. By the symmetry of T' and (52),

$$\langle u, T'f\rangle = \langle T'u, f\rangle = \langle u, g\rangle \text{ for all } u \text{ in } D'.$$

But D' is dense in H and so $T'f = g$. Hence $T'\colon D' \subseteq H \to H$ is self-adjoint

Claim 2: The operator $T\colon D \subseteq H \to H$ extends $T'\colon D' \subseteq H \to H$. Indeed, let v belong to D'. By the symmetry of T,

$$\langle Tu, Q_nv\rangle = \langle u, TQ_nv\rangle \text{ for all } u \text{ in } D \text{ and all } n.$$

Take limits as $n \to \infty$ to obtain

$$\langle Tu, v\rangle = \langle u, T'v\rangle \text{ for all } u \text{ in } D.$$

Since T is self-adjoint, v belongs to D and $Tu = T'v$. Thus $T\colon D \subseteq H \to H$ extends $T'\colon D' \subseteq H \to H$.

Since $T'\colon D' \subseteq H \to H$ is self-adjoint, it has no proper symmetric extensions. Therefore $D = D'$ and $T' = T$.

□

Theorem 23 (von Neumann)**.** *If $T\colon D \subseteq H \to H$ is self-adjoint, then the operator*

$$\mathrm{I} + T^2\colon D(T^2) \subseteq H \to H$$

is one-to one and onto, and its inverse $(\mathrm{I} + T^2)^{-1}$ is bounded, symmetric, and

$$0 \leq (\mathrm{I} + T^2)^{-1} \leq \mathrm{I}\,. \tag{53}$$

Furthermore, the operator $T \circ (\mathrm{I} + T^2)^{-1}\colon H \to H$ also is bounded.

Proof. To show that $\mathrm{I} + T^2$ maps $D(T^2)$ onto H, we examine the graph of T,

$$G(T) \equiv \{(u, Tu) \mid \; u \text{ in } D\} \subseteq H \oplus H.$$

As we already observed, since T is self-adjoint, $G(T)$ is a closed subspace of $H \oplus H$, considered as a Hilbert space with the natural Hermitian form making the decomposition orthogonal. Therefore, there is the following orthogonal decomposition of $H \oplus H$:

$$H \oplus H = G(T) \oplus G(T)^{\perp}. \tag{54}$$

We deduce from the self-adjointness of T that

$$G(T)^{\perp} = \{(-Tu, u) \mid \; u \text{ in } D\}\,.$$

Let h belong to H. According to (54), there are vectors u, v in D for which

$$(h,\, 0) = (u,\, Tu) + (-Tv,\, v),$$

that is,

$$h = u - Tv \text{ and } v = -Tu.$$

Hence u belongs to $D(T^2)$ and $h = u + T^2(u)$. Thus $\mathrm{I} + T^2\colon D(T^2) \to H$ is onto. Since T is symmetric

$$\langle (\mathrm{I} + T^2)u,\, u\rangle = \langle u,\, u\rangle + \langle Tu,\, Tu\rangle \geq \langle u,\, u\rangle \text{ for all } u \text{ in } D(T^2). \tag{55}$$

Thus $\mathrm{I} + T^2$ is one-to-one, (53) holds, and $(\mathrm{I} + T^2)^{-1}$ is symmetric since it is the inverse of a symmetric operator. We deduce from (55) and the Cauchy-Schwarz Inequality that $(\mathrm{I} + T^2)^{-1}$ is bounded.

Finally, we verify that $T \circ (\mathrm{I} + T^2)^{-1}\colon H \to H$ is bounded. Indeed, let h belong to H. Then

$$T \circ (\mathrm{I} + T^2)^{-1} h = Tv \text{ where } (\mathrm{I} + T^2)v = h.$$

By the symmetry of T,

$$\langle v,\, v\rangle + \langle Tv,\, Tv\rangle = \langle h,\, v\rangle,$$

from which we first deduce that $\|v\| \leq \|h\|$ and then that $\|Tv\| \leq \|h\|$. Hence

$$\|T \circ (\mathrm{I} + T^2)^{-1} h\| = \|Tv\| \leq \|h\| \text{ for all } h \text{ in } H,$$

and therefore $T \circ (\mathrm{I} + T^2)^{-1}$ is bounded. □

Theorem 24. *Let $T\colon D \subseteq H \to H$ be self-adjoint. Then T has a sequence of bounded approximations $\{T\colon H_n \to H_n\}$, which has the further property that if S in $\mathcal{L}(H)$ commutes with T, in the sense that $S \circ T = T \circ S$ on D, then S maps each H_k into H_k.*

Proof. The preceding theorem tells us that $(\mathrm{I}+T^2)^{-1}$ belongs to $\mathcal{L}(H)$ and is symmetric. Theorem 18 tells us that there is a unique spectral resolution $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ for $(\mathrm{I}+T^2)^{-1}$. Recall that any operator S in $\mathcal{L}(H)$ that commutes with $(\mathrm{I}+T^2)^{-1}$ also commutes with each projection $\mathcal{E}_\lambda$. For each natural number n, define

$$Q_n \equiv \mathrm{I}-\mathcal{E}_{1/(n+1)} \text{ and } H_n \equiv Q_n(H).$$

We deduce from Lemma 1 that $\{H_n\}_{n=1}^\infty$ is an ascending sequence of closed subspaces of H and each Q_n is the orthogonal projection of H onto H_n.

Claim 1: For each h in H, $\lim_{n\to\infty} Q_n h = h$. Indeed, by the right-continuity of $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbf{R}}$ to verify this claim is to show that $\mathcal{E}_0 = 0$. However, observe that, by (8), $\mathcal{E}_\lambda$ is equal to $\mathcal{E}_0$ for $\lambda \geq 0$ and equal to 0 for $\lambda < 0$. Therefore

$$(\mathrm{I}+T^2)^{-1}\mathcal{E}_0 = \int_{-1}^{1} \lambda d\,\mathcal{E}_\lambda\mathcal{E}_0 = 0.$$

Since $(\mathrm{I}+T^2)^{-1}$ is invertible, $\mathcal{E}_0 = 0$.

Claim 2: We claim that $\{H_k\}_{k=1}^\infty$ reduces T. Indeed, fix a natural number k. As noted above, $(\mathrm{I}+T^2)^{-1}$ commutes with each $\mathcal{E}_\lambda$ and therefore maps H_k into itself. Moreover, by (14),

$$\langle(\mathrm{I}+T^2)^{-1}h,\, h\rangle = \int_{1/k}^{1} \lambda d\,\langle\mathcal{E}_\lambda h,\, h\rangle \geq 1/k\|h\|^2 \text{ for all } h \text{ in } H_k$$

In particular, $(\mathrm{I}+T^2)^{-1}\colon H_k \to H_k$ is positive definite, and consequently, by Proposition 6, is invertible. On the other hand, the bounded linear operator $T(\mathrm{I}+T^2)^{-1}$ commutes with $(\mathrm{I}+T^2)^{-1}$. By the commutativity property of the spectral resolution, $T(\mathrm{I}+T^2)^{-1}$ commutes with each $\mathcal{E}_\lambda$, and so $[T(\mathrm{I}+T^2)^{-1}](H_k) \subseteq H_k$. Therefore

$$T(H_k) = T(\mathrm{I}+T^2)^{-1}(H_k) \subseteq H_k.$$

Claim 3: We claim that the commutativity property holds. Indeed, suppose S in $\mathcal{L}(H)$ commutes with T. Then S also commutes with $(\mathrm{I}+T^2)^{-1}$. By the commutativity property of the spectral resolution, S commutes with each $\mathcal{E}_\lambda$ and therefore maps each subspace H_k into itself.

□

3. An Extra Lemma

Lemma 25 (The Approximation Lemma)**.** *Let $T\colon D \subseteq H \to H$ be self adjoint. For each n, let H_n be a closed subspace of D for which $H_n \subseteq H_{n+1}$ and $T(H_n) \subseteq H_n$, and let Q_n be the orthogonal projection of H onto H_n. Suppose that for each h in H, $\lim_{n\to\infty} Q_n h = h$. Then*

$$Th = \lim_{n\to\infty} TQ_n h \text{ for all } h \text{ in } D.$$

Proof. Let h belong to D. For each n, $TQ_n h$ and $T(TQ_n h)$ belong to $H_n \subseteq D$. Thus

$$||TQ_n h||^2 = \langle TQ_n h,\, TQ_n h\rangle = \langle T(TQ_n h),\, Q_n h\rangle = \langle T(TQ_n h),\, h\rangle = \langle TQ_n h,\, Th\rangle \text{ for all } n,$$

so that, by the Cauchy-Schwarz Inequality, $||TQ_nh|| \leq ||Th||$ for all n. Thus the sequence $\{TQ_nh\}_{n=1}^{\infty}$ is bounded. Since the symmetric operator T maps Q_{n+k} into itself and Q_n into itself, for each n and k,

$$\|TQ_{n+k}h\|^2 - \|TQ_nh\|^2 = \|TQ_{n+k}h - TQ_nh\|^2. \tag{56}$$

Thus the sequence $\{\|TQ_nh\|\}_{n=1}^{\infty}$ is increasing, and since it is bounded, it converges. The completeness of H, together with (56), tells us that $\{TQ_nh\}_{n=1}^{\infty}$ converges: call the limit f. We claim that $f = Th$. Indeed, for each n,

$$\langle TQ_nh, u\rangle = \langle Q_nh, Tu\rangle \text{ for all } u \text{ in } D$$

Take the limit as $n \to \infty$ to see that

$$\langle z, u\rangle = \langle h, Tu\rangle \text{ for all } u \text{ in } D.$$

The self-adjointness of T tells us that $z = Th$. □

References

[1] J.L. Doob and B.G. Koopman, On analytic functions with positive imaginary part, Bull AMS, vol 40 (1934), pp. 601-605
[2] Nelson Dunford and Jacob T. Schwartz, Linear Operators, Wiley Classics Library, 1989
[3] Paul Halmos, What does the spectral theorem say?, Am. Math Monthly, vol 70 (1963), pp. 241-247
[4] Peter Lax , *Functional Analysis*, Pure and Applied Mathematics, Wiley Interscience, 2002
[5] John von Neumann, Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren, Math. Ann., vol 102 (1929), pp. 49-131
[6] John von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton Landmarks in Mathematics, Princeton University Press, 1955: (reprint of the 1932 book)
[7] Michael Reed and Barry Simon, *Methods of Modern Mathematical Physics*, Academic Press (1972)
[8] Frederick Riesz, Über die linearen Transformationen des komplexen Hilbertschen Raumes, Acta Litterumm Scientarum (Szegreb) vol 5 (1930), pp. 23-54
[9] Frederick Riesz and E. R. Lorch, The integral representation of unbounded self-adjoint operators in a Hilbert space, Trans. AMS, vol 39 (1936), pp. 331-340
[10] Frederick Riesz and Béla Sz. Nagy, *Functional Analysis*, Dover Books in Mathematics, 1990
[11] H. L. Royden and Patrick M. Fitzpatrick, *Real Analysis*, 4th Edition, Prentice Hall, 2010
[12] Marshall H. Stone, Linear Transformations in Hilbert Space, Proc. Nat. Acad. Sci., vol 15 (1929), pp. 198-200
[13] Augus E. Taylor and David Lay, *Introduction to Functional Analysis*, Kreiger, 1986
[14] Richard L. Wheedon and Antoni Zygmund, Integration and Measure, Pure and Applied Mathematics, Marcel Dekker, 1977

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