MATH 141 – CALCULUS II
FIRST MIDTERM EXAM SOLUTIONS

(1) Note that the graphs intersect at $x = 0$ and $x = 1$. Then

$$\begin{aligned}
\text{Volume} &= \int_0^1 2\pi x(\sqrt{x} - x^2)\, dx \\
&= 2\pi \int_0^1 (x^{3/2} - x^3)\, dx \\
&= 2\pi \left[\frac{2}{5}x^{5/2} - \frac{x^4}{4}\right]_0^1 \\
&= 3\pi/10
\end{aligned}$$

(2) Let h be the depth into the tank. Then at a given h the infinitesimal volume of a slice is $\pi(4 - h^2)\Delta h$. The distance to the rim is h. Hence

$$\begin{aligned}
\text{Work} &= \int_0^2 \pi(4 - h^2)h\, dh \\
&= \pi\left[2h^2 - h^4/4\right]_0^2 \\
&= 4\pi
\end{aligned}$$

(3) The intersection points occur when $3 - x^2 = x^2 + 1$, or $x = \pm 1$. Hence, the area is given by

$$\begin{aligned}
\text{Area} &= \int_{-1}^1 (3 - x^2 - (x^2 + 1))dx = \int_{-1}^1 (2 - 2x^2)dx \\
&= (2x - 2x^3/3)\Big]_{-1}^1 = 8/3
\end{aligned}$$

For the center of mass, notice that the region is symmetric with respect to $x \mapsto -x$. Hence, $\bar{x} = 0$. For $\bar{y}$, we need to calculate

$$\begin{aligned}
M_x &= \frac{1}{2}\int_{-1}^1 ((3 - x^2)^2 - (x^2 + 1)^2)dx \\
&= \frac{1}{2}\int_{-1}^1 (8 - 8x^2)dx \\
&= 4(x - x^3/3)\Big]_{-1}^1 = 16/3
\end{aligned}$$

Hence, $\bar{y} = (16/3)/(8/3) = 2$. Alternatively, notice that the region is symmetric about the axis $y = 2$! Indeed, $f(x) - 2 = 1 - x^2 = -(x^2 - 1) = -(g(x) - 2)$.

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(4) We have $\dot{x} = \sin t$, $\dot{y} = 1 - \cos t$, so

$$\dot{x}^2 + \dot{y}^2 = \sin^2 t + (1 - \cos t)^2 = 2 - 2\cos t = 4\sin^2(t/2)$$

Hence, the length is

$$\int_0^\pi \sqrt{\dot{x}^2 + \dot{y}^2} dt = \int_0^\pi 2\sin(t/2) = -4\cos(t/2)]_0^\pi = 4$$