MATH 141 – CALCULUS II
SECOND MIDTERM EXAM SOLUTIONS

(1) (a) A right triangle with angle θ, opposite side length $\sqrt{2}$ and adjacent side length 1 satisfies $\tan\theta = \sqrt{2}$. Since the hypotenuse has length $\sqrt{3} = \sqrt{1^2 + (\sqrt{2})^2}$, $\cos\theta = 1/\sqrt{3}$.

(b) $f'(x) = -3/x^4 - 3 < 0$, so by the Mean Value Theorem f must be one-to-one.

(c) Notice that $f^{-1}(-2) = 1$. Hence,

$$(f^{-1})'(-2) = \frac{1}{f'(f^{-1}(-2))} = \frac{1}{f'(1)} = -\frac{1}{6}$$

(2) (a) $d(3^{x^2})/dx = 3^{x^2}(d/dx)\ln 3^{x^2} = 3^{x^2}(d/dx)(x^2 \ln 3) = 3^{x^2}(2x\ln 3)$.

(b) Notice that

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

Now if $u = 3x/2$, we have

$$\begin{aligned}
\int_0^{1/\sqrt{3}} \frac{dx}{\sqrt{4-9x^2}} &= \frac{1}{2}\int_0^{1/\sqrt{3}} \frac{dx}{\sqrt{1-(3x/2)^2}} \\
&= \frac{1}{3}\int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1-u^2}} \\
&= \frac{1}{3}(\sin^{-1}(\sqrt{3}/2) - \sin^{-1}(0)) \\
&= \pi/9
\end{aligned}$$

(3) (a) We have by L'Hôpital's rule:

$$\begin{aligned}
\ln\left(\lim_{x\to\infty}\left(1+\frac{1}{3x}\right)^{2x}\right) &= \lim_{x\to\infty} \ln\left(1+\frac{1}{3x}\right)^{2x} \\
&= \lim_{x\to\infty} 2x\ln\left(1+\frac{1}{3x}\right) \\
&= 2\lim_{x\to\infty} \frac{\ln(1+1/3x)}{1/x} \\
&= 2\lim_{x\to\infty} \frac{1}{1+1/3x}\left(\frac{-1/3x^2}{-1/x^2}\right) \\
&= 2/3
\end{aligned}$$

So the limit in the problem is $e^{2/3}$.

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(4) By L'Hôpital's rule applied twice:

$$\lim_{x\to 1} \frac{\ln x - x + 1}{x^3 - 3x + 2} = \lim_{x\to 1} \frac{1/x - 1}{3x^2 - 3} = \lim_{x\to 1} \frac{-1/x^2}{6x} = -1/6$$

(5) Use an integrating factor, or simply observe that

$$\begin{aligned} x^5 \frac{d}{dx}(x^{-4}y) &= xy' - 4y = x^2 \\ \frac{d}{dx}(x^{-4}y) &= x^{-3} \\ x^{-4}y &= -x^{-2}/2 + C \\ y &= -x^2/2 + Cx^4 \end{aligned}$$

Plugging in $y(1) = 1$, we find $C = 3/2$.