MATH 141 – CALCULUS II
MIDTERM EXAM # 4 SOLUTIONS

(1) (a) Since $na_n \to 2$, we may assume $1 \leq na_n \leq 3$, or $1/n \leq a_n$ and $a_n^2 \leq 9/n^2$. The first inequality implies that $\sum a_n$ diverges (comparison with the harmonic series) and the second inequality implies $\sum a_n^2$ converges (comparison with the p-series, $p = 2$). (b) This is a geometric series with $a = 1/3$ and $r = -2/3$. Hence, the sum is

$$\frac{a}{1-r} = \frac{1/3}{1+2/3} = \frac{1}{5}$$

(c) This is an alternating series with

$$a_n = \frac{n+1}{n^2+n+1}$$

Since $a_n \to 0$ (by L'Hôpital's rule), it suffices to show $a_n \geq a_{n+1}$ for n sufficiently large. This follows directly by algebra, or notice that $f'(x) < 0$ (and so f is decreasing) for x large, where $f(x) = (x+1)/(x^2+x+1)$.

(2) (a) By the ratio test

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{2n+2}(2n)!}{(2n+2)!n^{2n}} = \frac{(1+1/n)^{2n+2}}{(2+2/n)(2+1/n)} \to \frac{e^2}{4}$$

So the radius of convergence is $4/e^2$. (b) By the root test,

$$\sqrt[n]{|a_n|} = \frac{(\ln n)^{3/n}}{n^{2/n}} \to 1$$

(note that since $\ln(n)/n \to 0$ and $\ln(\ln n)/n \to 0$ by l'Hôpital's rule, we have $n^{1/n} = \exp(\ln n/n) \to 1$ and $(\ln n)^{1/n} = \exp(\ln(\ln n)/n) \to 1$). So the radius of convergence in this case is 1.

(3) For (a),

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$
$$e^x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!}$$

Date: April 29, 2011.

For (b),

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$$

$$\frac{1}{2-t^2} = \frac{1}{2}\frac{1}{(1-t^2/2)} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{t^{2n}}{2^n}$$

$$\int_0^x \frac{dt}{2-t^2} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2^n(2n+1)}$$

(4) The n-th coefficient in the expansion is $f^{(n)}(0)/n!$. So

$$\frac{f^{(10)}}{10!} = \frac{-1}{4^5(5!)^2} \qquad f^{(10)}(0) = -\frac{10!}{4^5(5!)^2}$$