MATH 241 – CALCULUS III
SECOND MIDTERM EXAM SOLUTIONS

(1) For (a), $\nabla f = 3ye^{3x}\,\hat{\mathbf{i}} + e^{3x}\,\hat{\mathbf{j}}$, $\nabla f(0,1) = 3\,\hat{\mathbf{i}} + \hat{\mathbf{j}}$, and so $\mathbf{u} = 3/\sqrt{10}\,\hat{\mathbf{i}} + 1/\sqrt{10}\,\hat{\mathbf{j}}$. For (b), set $g(x,y,z) = 3e^x + 2xz + \sin(x+2y) - 3z$. Then

$$\nabla g = (3e^x + 2z + \cos x + 2y)\,\hat{\mathbf{i}} + 2\cos(x+2y)\,\hat{\mathbf{j}} + (2x-3)\,\hat{\mathbf{k}}$$
$$\nabla g(0,0,1) = 6\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} - 3\,\hat{\mathbf{k}}$$

So the equation of the plane is: $6x + 2y - 3z + 3 = 0$.

(2) For (a), the limit doesn't exist, since if $x = y$ the function is $= 2$ whereas for $x = -y$ the function $= 0$. For (b), differentiate both sides of the equation in part 1(b) with respect to z:

$$3e^x\frac{\partial x}{\partial z} + 2z\frac{\partial x}{\partial z} + 2x + \cos(x+2y)\frac{\partial x}{\partial z} = 3$$

Evaluating at the point $(0,0,1)$ we find $\partial x/\partial z = 1/2$.

(3) By definition: $\mathbf{N}(s) = \mathbf{T}'(s)/\|\mathbf{T}'(s)\|$. Since $\mathbf{T}(s)$ is a unit vector

$$0 = \frac{d}{ds}(\mathbf{T}\cdot\mathbf{T}) = 2\mathbf{T}\cdot\mathbf{T}'$$

So in particular, $\mathbf{T}\cdot\mathbf{N} = 0$. For (b), notice that from the definition of the binormal, $\mathbf{B}\cdot\mathbf{N} = 0$ for all s. Differentiate this to find $\mathbf{B}\cdot\mathbf{N}' = -\mathbf{B}'\cdot\mathbf{N}$. The result follows from the definition of torsion and the fact that $\mathbf{N}$ is a unit vector.

(4) Compute the partials

$$f_x = 6x - 3y^2 \qquad f_y = -6xy + 3y^2 + 6y$$

Setting the first equal to zero we have $x = y^2/2$. Substituting this into the second, we have $0 = -3y^3 + 3y^2 + 6y = -3y(y+1)(y-2)$. The critical points are therefore $(0,0)$, $(1/2,-1)$, and $(2,2)$. The second derivatives are $f_{xx} = 6$, $f_{yy} = -6x + 6y + 6$, $f_{xy} = -6y$, so the Hessians at the critical points are

$$\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}, \quad \begin{pmatrix} 6 & 6 \\ 6 & -3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 6 & -12 \\ -12 & 6 \end{pmatrix}$$

respectively. By the second derivative test, $(0,0)$ is a local minimum and the other two critical points are saddles.

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